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FIRST PRINCIPLES

OF THE

DIFFERENTIAL AND INTEGRAL CALCULUS,

OR THE

DOCTRINE OF FLUXIONS,

INTENDED

AS AN INTRODUCTION TO THE PHYSICO-MATHEMATICAL SCIENCES;

TAKEN CHIEFLY

FROM THE MATHEMATICS OF BÉZOUT, (Étienne)

AND TRANSLATED FROM THE FRENCH

FOR THE USE OF THE STUDENTS OF THE UNIVERSITY

AT

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TO THE FIRST EDITION.

The following treatise, except the introduction and notes, is a translation of the *Principes de Calcul qui servent d'Introduction aux Sciences Physico-Mathématiques* of Bézout. It was selected on account of the plain and perspicuous manner for which the author is so well known, as also on account of its brevity and adaptation in other respects to the wants of those who have but little time to devote to such studies. The easier and more important parts are distinguished from those which are more difficult or of less frequent use, by being printed in a larger character. In the Introduction, taken from Carnot's *Reflexions sur la Metaphysique du Calcul Infinitesimal*, a few examples are given to show the truth of the infinitesimal method, independently of its technical form. Moreover in the 4th of the notes, subjoined at the end, some account is given from the same work, of the methods previously in use, analogous to the Infinitesimal Analysis. The other notes are intended to supply the deficiencies of Lacroix's Algebra (Cambridge Translation), considered as a preparatory work.

Since this treatise was announced, the compiler of the Cambridge Mathematics has been obliged, on account of absence from the country and infirmity of sight, to resign his work into other hands. This circumstance is mentioned to account for the delay attending the publication, as well as the occasional want of conformity to other parts of the course in the mode of rendering certain words and phrases which a revision of the translation, had it been practicable, would have easily remedied.

Cambridge, July, 1824.

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INTRODUCTION.

The Infinitesimal Analysis, as presented in the following Treatise, proposes to ascertain the relation of definite, assignable quantities, by comparing them with quantities which are here called *infinitely small.* But by *infinitely small* quantities is meant *quantities which may be made as small as we please, without altering the value of those with which they are compared, and whose ratio is sought.* The first idea of this calculus was probably suggested by the difficulties which are often met with in endeavouring to express by equations the different conditions of a problem, and in resolving these equations when formed. When the exact solution of a problem is too difficult, it is natural to endeavour to approximate as nearly as possible to an accurate solution, by neglecting those quantities which embarrass the combinations, if it is seen that they are so small, that the neglect of them will not materially affect the result. Thus, for example, it being found very difficult to discover directly the properties of curves, mathematicians would have recourse to the expedient of considering them as polygons of a great number of sides. For, if a regular polygon be inscribed in a circle, it is manifest, that these two figures, although they can never coincide and become the same, approach each other the more nearly in proportion as the number of the sides of the polygon increases. Whence it follows, that, by supposing the number of sides very great indeed, we may, without any very sensible error, attribute to the circle the properties which are found to belong to the inscribed polygon. And if, in the course of a calculation, we should find a circumstance in which the process would be much simplified by neglecting one of these exceedingly small sides, when compared with a radius, for example, we might evidently do it without inconvenience, since

the error which would result would be so extremely small, that it need not be noticed.

Let it be proposed, for example, to draw a tangent to the point M of the curve AMB (*fig.* 1) considered as part of the circumference of a circle.

Let Q be the centre, and APQ the axis; call the absciss AP, x, and the corresponding ordinate PM, y, and let TP be the subtangent sought.

To find this, we consider the circle as a polygon of a very great number of sides, and Mm as one of these sides; we produce Mm until it meets the axis, and it is evidently the tangent in question, since it does not penetrate the polygon. We let fall upon AQ the perpendicular mp, and call the radius of the circle, a. The similar triangles Mrm, TPM, give

$$Mr : rm :: TP : PM, \text{ or } \frac{Mr}{rm} = \frac{TP}{y}.$$

Now, since the equation of the curve for the point M is

$$y^2 = 2ax - x^2, \text{ (Trig. 101,)}$$

it will be, for the point m,

$$(y + rm)^2 = 2a(x + Mr) - (x + Mr)^2,$$

developing, we have

$$y^2 + 2y.\overline{rm} + \overline{rm}^2 = 2ax + 2a.\overline{Mr} - x^2 - 2x.\overline{Mr} - \overline{Mr}^2,$$

from which, if we subtract the equation for the point M, we have

$$2y.\overline{rm} + \overline{rm}^2 = 2a.\overline{Mr} - 2x.\overline{Mr} - \overline{Mr}^2.$$

Whence, by reducing,

$$\frac{Mr}{rm} = \frac{2y + rm}{2a - 2x - Mr}.$$

Substituting for $\frac{Mr}{rm}$ its value found above, and multiplying by y, we have

$$TP = \frac{y(2y + rm)}{2a - 2x - Mr}.$$

If now rm and Mr were known, we should have the value of TP sought; they are, however, very small, since they are each less than Mm, which is itself, by supposition, very small. They are, moreover, perfectly arbitrary, since there is nothing in the supposition to limit their magnitude, and they may be rendered indefinitely small without affecting the lines TP and PM, with

which they are compared. We may therefore neglect, without any material error, these quantities so small, compared with the quantities $2y$ and $2a - 2x$, to which they are joined; and the equation is thus reduced to $TP = \frac{y^2}{a - x}$. This value, thus found by neglecting the very small quantities mr and Mr, is not only nearly accurate, as might be supposed, but absolutely exact, as is thus shown. The similar triangles QPM, MPT, give

$$QP : MP :: MP : PT;$$

whence

$$PT = \frac{\overline{MP}^2}{QP} = \frac{y^2}{a - x}, \text{ as above.}$$

The above result is thus obtained by a balance of errors; the error made in supposing the circumference to be a polygon being compensated by neglecting the small quantities mr and Mr to obtain the final value: and the omission of these quantities is not only allowable, but is absolutely necessary to fulfil the conditions of the problem.†

As a second example, we suppose that it is required to find the surface of a given circle.

We here also may consider the curve as a regular polygon of a great number of sides. The area of a regular polygon is equal to the product of its perimeter, by half of the perpendicular let fall from the centre upon one of the sides. Therefore the circle, considered as a polygon of a great number of sides, is equal to the product of its circumference by half the radius; a proposition which is no less exact than the result found above.

In the examples just given, it is seen that great advantage is obtained by employing quantities which are *very small* compared with the principal quantities in question. The same principle once admitted may be very generally applied; all other curves may, as well as the circle, be considered as polygons of a

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† If it be asked, how we may be sure in similar cases, that the compensation of errors has taken place, it may be observed, that the error, if any exist, depends upon the arbitrary quantities rm and Mr, and may be made as small as we please by diminishing these quantities; but as these disappear in the final result, the error disappears with them, and leaves the result perfectly accurate.

great number of sides. All surfaces may be considered as divided into a multitude of zones, all bodies into corpuscles; in short, all quantities to be decomposed into small parts of the same kind as themselves. Hence will arise many new relations and combinations, and it is easy to judge, from the examples given above, of the resources furnished to the calculus by the introduction of these elementary quantities.

The advantage obtained is even much greater than we should at first expect, since, in many cases, as we have seen above, the method employed is not merely an approximation, but leads to perfectly accurate results. It becomes therefore an interesting object, to ascertain when this is the case, to extend the application of the principle, and to reduce the methods employed to a strict and regular system. Such is the object of the infinitesimal analysis.

We shall now give some problems, tending to throw light on the mode of reasoning employed in this calculus.

1. *To draw a tangent to the common cycloid.*

Let AEB (*fig.* 54) be a common cycloid, of which the generating circle is $E\,p\,q\,F$. The principal property of this cycloid is, that for any point m, the portion $m\,p$ of the ordinate, comprised between the curve and the circumference of the generating circle, is equal to the arc $E\,p$ of that circumference.

Draw to the point p of this circumference a tangent $p\,T$, and let it be required to find the point T where this tangent is intersected by $m\,T$, the tangent of the cycloid.

In order to this, we draw a new ordinate $n\,q$ infinitely near to the first $m\,p$, and through m draw $m\,r$ parallel to the little arc $p\,q$, which, as well as $m\,n$, we consider as a straight line.

It is evident that the two triangles $m\,n\,r$, $T\,m\,p$, will be similar, and we shall consequently have $m\,r : n\,r :: T\,p : m\,p$. But since, by the properties of the cycloid, we have $E\,q = n\,q$ and $E\,p = m\,p$, we shall have, by subtracting the second of these equations from the other, $E\,q - E\,p = n\,q - m\,p$, or $p\,q = n\,r$, or $m\,r = n\,r$. Wherefore, by reason of the proportion found above, we have $T\,p = m\,p$, or $T\,p = E\,p$, that is, the subtangent $T\,p$ is always equal to the corresponding arc $E\,p$. This equation

is disengaged, by the disappearance of $m\,r$ and $n\,r$, from every consideration of infinite or arbitrary; whence the proposition is rigorously and necessarily exact.

2. *To show that in motion uniformly accelerated, the spaces described are as the squares of the times, reckoning from the beginning of the motion.*

In this motion, the accelerating force acts constantly in the same manner, wherefore, if we suppose g to be the velocity communicated in each unit of time, the successive velocities will evidently form the series g, $2\,g$, $3\,g$, $4\,g$, &c.; so that after a number of units of time marked by t, the velocity acquired will be as many times g as there are units in t, that is, calling the velocity u, u will equal $g\,t$.

Since the velocities g, $2\,g$, &c. are each nothing but the space which the moving body describes in the corresponding interval of time, the total space described during the time t will be the sum of the terms of this arithmetical progression. But the sum of the terms of such a progression is found by multiplying the sum of the first and last terms by half the number of terms. Whence, this sum will be, (substituting u for its value $g\,t$, which is the last term,) $(g+u)\times\frac{t}{2}$. Whence, if we represent the space by s, we have

$$s=(g+u)\times\frac{t}{2}.$$

Let us now conceive that the accelerating force acts without interruption, or, which is the same, that the time is divided into an infinite number of infinitely small parts called *instants*, and that, at the beginning or end of each instant, the accelerating force gives a new impulse to the moving body. We conceive, moreover, that it acts by infinitely small degrees. Then g, being infinitely small compared with u, which is the velocity acquired in the infinite number of instants indicated by t, we must, in the equation $s=(g+u)\frac{t}{2}$, omit g, and we shall have $s=\frac{u\,t}{2}$.

If we call the velocity acquired at the end of a second, p, the velocity acquired after a number, t, of seconds, will be $t\,p$. Whence $u=p\,t$. The equation $s=\frac{u\,t}{2}$, found above, will thus

become $s = \frac{p\,t^2}{2}$. If, therefore, we represent by S another space described in the same manner during the time T, we shall in like manner have $S = \frac{p\,T^2}{2}$, whence we may conclude

$$s : S :: \frac{p\,t^2}{2} : \frac{p\,T^2}{2} :: t^2 : T^2,$$

which was to be proved; and which, being freed from all consideration of infinite, is necessarily and rigorously exact.

3. *To determine in what manner to divide a quantity,* a, *into two parts, in such a manner that the product of these parts shall be the greatest possible.*

Let x be one of the parts, the other will be $a - x$, and the product will be $a\,x - x^2$. Let this product be supposed the greatest possible product of the two parts of a. Suppose x to take a new value infinitely little differing from its present value. Let this be $x + h$. This value, substituted in the above product, gives $a\,x + a\,h - x^2 - 2\,x\,h - h^2$. If we subtract the former value $a\,x - x^2$ from this, there will remain $a\,h - 2\,h\,x - h^2$. Now since $a\,x - x^2$ was by supposition the greatest product possible, this increase must be nothing; therefore

$$a\,h - 2\,h\,x - h^2 = 0; \text{ or } a\,h = 2\,h\,x + h^2.$$

But h^2 is infinitely small compared with $2\,h\,x$, since it is an infinitely small part of an infinitely small quantity,† and may therefore be neglected. We therefore have

$$a\,h = 2\,h\,x, \text{ or } a = 2\,x, \text{ or } x = \frac{a}{2},$$

on the supposition that the product is the greatest possible. Whence we conclude, that each of the two parts is one half of a.

These examples are introduced to show how the principles of the infinitesimal analysis may be employed in ordinary reasoning and in common algebra. In the following treatise the same principles are reduced to a system in the Differential and Integral Calculus.

† If h is only $\frac{1}{1{,}000{,}000}$, $h^2 = \frac{1}{1{,}000{,}000{,}000{,}000}$.

PRINCIPLES

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Preliminary Principles.

Algebra and the application of Algebra to Geometry contain the rules necessary to calculate quantities of any definite magnitude whatever. But quantities are sometimes considered as *varying* in magnitude, or as having arrived at a given state of magnitude by different, successive variations. The consideration of these variations gives rise to another branch of analysis, which is of the greatest use in the physico-mathematical sciences, and especially in Mechanics, in which we often have no other means of determining the ratio of quantities, which enter into questions relative to this science, than that of considering the ratios of their variations, that is to say, of the increments and decrements which they each instant receive.

As an introduction, therefore, to Mechanics and the other branches of Natural Philosophy, it is well to obtain some knowledge of this part of the calculus, the object of which is, to decompose quantities into the elements of which they are composed, and to ascend or go back again from the elements to the quantities themselves. This is, strictly speaking, rather an application of the methods, and even a simplification of the rules of the former branches of analysis, than a new branch.

2. We propose to ourselves two objects. The first is, to show how to descend from quantities to their elements; and the method of accomplishing this, is called the *Differential Calculus*. The

second is, to point out the way of ascending from the elements of quantities to the quantities themselves; and this method is called the *Integral Calculus.*

So much of these two methods, as is of essential importance, may be easily understood, as it is but a consequence of former parts of analysis. Those branches of these methods, which require more delicate researches, or which are of a less frequent application, will be distinguished by being printed in a smaller type.

3. As we are about to consider quantities with relation to their elements, that is to say, their infinitely small increments, it is necessary, before proceeding farther, to explain what is meant by quantities infinitely small, infinitely great, &c., and to point out the subordination which must be established between these quantities in calculation.

4. We say that a quantity is infinitely great or infinitely small with regard to another, when it is not possible to assign any quantity sufficiently large or sufficiently small to express the ratio of the two, that is, the number of times that one contains the other.

Since a quantity, as long as it is such, must always be susceptible of increase and diminution, there can be no quantity so small or so great, with regard to another quantity, but we may conceive of a third infinitely smaller or greater. For example, if x is infinitely great with regard to a, although it be then impossible to assign their ratio, this does not prevent our conceiving of a third quantity, which shall be to x, as x is to a, that is, which shall be the fourth term of a proportion, of which the three first are $a : x :: x :$; this fourth term, which is $\frac{x^2}{a}$, must therefore be infinitely greater than x, since it contains x as many times as x is supposed to contain a. In the same manner, nothing prevents our conceiving of the fourth term of this proportion, $x : a :: a : x$; and this fourth term, which is $\frac{a^2}{x}$, will be infinitely smaller than a, since it is contained by a as many times as a is supposed to be contained by x. There are no bounds to the imagination in this respect; and we may still conceive of a new quantity, which shall be infinitely smaller with regard to $\frac{a^2}{x}$ than $\frac{a^2}{x}$ is with re-

gard to a. We call these quantities *infinitely great* or *infinitely small quantities* of *different orders.*

In general, the product of two infinitely great quantities or of two infinitely small quantities of the *first order*, is infinitely greater or infinitely smaller than either of the two factors; for, $x\,y : y :: x : 1$; but, if x is infinite, it contains unity an infinite number of times, $x\,y$, therefore, contains y an infinite number of times. A similar course of reasoning shows that a product or a power, of any number of dimensions whatever, and all of whose factors are infinite, is of an order of infinite marked by the number of its factors; thus, when x is infinite, x^4 is an infinite of the fourth order, that is, infinitely greater than x^3, which is infinitely greater than x^2, which is itself infinitely greater than x. For

$$x^4 : x^3 :: x^3 : x^2 :: x^2 : x :: x : 1.$$

On the contrary, if x were infinitely small, x^4 would be then an infinitely small quantity of the fourth order, that is, infinitely smaller than x^3, while x^3 would be infinitely smaller than x^2, which would be infinitely smaller than x.

Again, a fraction, whose numerator is a finite quantity, and whose denominator is any power of an infinite quantity, is of an order of infinitely small quantities, marked by the exponent of that power. Thus, $\frac{b}{x^2}$, for example, is an infinitely small quantity of the second order, if x is infinite; and $\frac{b}{x^3}$ an infinitely small quantity of the third order. For

$$\frac{b}{x^2} : \frac{b}{x} :: \frac{1}{x} : 1 :: 1 : x.$$

But if a product have not all its factors infinite, then its order of infinity is to be determined by the number of those factors only which are infinite; thus $a\,x\,y$, for example, is of the same order as $x\,y$; since $a\,x\,y : x\,y :: a : 1$, and this last ratio, $a : 1$ or $\frac{a}{1}$, is a determinate ratio, if a is a finite quantity.

This difference is worthy of observation in the comparison of infinitely great or infinitely small quantities with each other or with other quantities, with regard to which they are infinitely great or small. If x be infinite with regard to a, nothing can measure their ratio; but, on the same supposition, the ratio of x, to x mul-

tiplied or divided by any finite number whatever, is a finite ratio. For example, x being infinite or infinitely small, is not comparable to a, a being supposed a finite number; but it may be compared with $a\,x$, since $x : a\,x :: 1 : a$.

5. To express, by the calculus, that a quantity x is infinite with regard to another quantity a; or, which is the same thing, to express that a is infinitely small with regard to x, we must, in the algebraical expression where these quantities are found together, reject all the powers of x lower than the highest, and consequently all those terms without x. If, for example, in $\frac{3\,x+a}{5\,x+b}$, x is supposed infinite with regard to a and b, we must suppress a and b, and we shall have $\frac{3\,x}{5\,x}$ or $\frac{3}{5}$ for the value of $\frac{3\,x+a}{5\,x+b}$, when x is infinite. For $\frac{3\,x+a}{5\,x+b}$ is the same thing as $\frac{3+\frac{a}{x}}{5+\frac{b}{x}}$, dividing the numerator and denominator by x; and, when we suppose x to be infinite with regard to a and b, the fractions $\frac{a}{x}$ and $\frac{b}{x}$, which represent the ratios of a and b to x, must necessarily be suppressed, since, by this supposition, these ratios are less than any quantity whatever; wherefore, in this case, the proposed quantity is reduced to $\frac{3}{5}$.

The quantity $x^2 + a\,x + b$ would, in like manner, be reduced to x^2, on the supposition that x were infinite. For it is only on the supposition that b adds nothing to the value of $a\,x + b$ that x can be said to be infinite: and in the same manner, it is only by supposing that $a\,x$ adds nothing to the value of $x^2 + a\,x$ that x^2 can be said to be infinite. Wherefore, both $a\,x$ and b must be considered as of no value by the side of x^2, and are therefore to be rejected, and the quantity is reduced to x^2.

If, on the contrary, x were infinitely small, it would be necessary to retain those terms only in which the exponent of x is smallest. Thus $x^2 + a\,x$ is reduced to $a\,x$, when x is infinitely small; and $\frac{a\,x+b}{c\,x+d}$, upon the same supposition, is reduced to $\frac{b}{d}$.

It need not be apprehended that these omissions will affect the consequences to be drawn from the calculations in which they may be made. On the contrary, it is only by these omissions that we can express what we mean to express, viz. that x is infinitely great or infinitely small. It is only by these omissions that we can arrive at a conclusion conformable to the supposition which we have made. For if, when we supposed x infinite, we should not reject the terms just pointed out; if, for example, in $\frac{3x+a}{5x+b}$ or $\frac{3+\frac{a}{x}}{5+\frac{b}{x}}$, we should not reject $\frac{a}{x}$ and $\frac{b}{x}$, then the calculus, not expressing that $\frac{a}{x}$ and $\frac{b}{x}$ are ratios less than any assignable quantity, would not answer what it is required to know, viz. what is the value of that quantity when x is infinite; in short by allowing $\frac{a}{x}$ and $\frac{b}{x}$ to have any effect on the value sought, we contradict the supposition which we have made, that x is infinite.

We shall not want occasions for verifying the exactness of this principle of *neglecting infinite quantities of the inferior orders.* For the present, the following example will confirm the reasoning just made use of. Let there be the series $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, &c.; the terms of this series evidently approach nearer and nearer to unity, yet without ever being able to pass this limit. Now each term may be represented by $\frac{x}{x+1}$, by substituting for x the number expressing the place of that term. Since then the terms continually approach unity, and that the more nearly, as they are farther from the beginning of the series, they can reach that limit only at an infinite distance from the beginning of the series; in order, therefore, to express the last term of this series, we must suppose in $\frac{x}{x+1}$ that x is infinite; but, conformably to the principle, this quantity must then be reduced to $\frac{x}{x}$, that is to say, to 1; the omission, therefore, of the term $+1$ in the expression $\frac{x}{x+1}$, so far from making the conclusion false, is, on the contrary, that which makes it what it ought to be. In short, by making this

omission, we act conformably to the supposition which has been made.

Such is the subordination which must be established in the calculus, between infinitely great and infinitely small *quantities* of different orders. But in the application of this principle of the omission of quantities, certain particular cases may occur which it will be well to notice.

Suppose we have the two quantities

$$x^2 + a x + b, \text{ and } x^2 + a x + c;$$

when x is infinite, each of these is evidently reduced to x^2, so that their difference, in this case, seems to be nothing. But if we take this difference according to the common rules, we find it $b - c$, or $c - b$, whether x be infinite or not. This seeming difficulty, however, is easily solved. For the difference of these quantities is really $b - c$ or $c - b$; but when we seek this difference, after having supposed x infinite in each, it is the same as asking what this difference is, compared to the quantities themselves; and, since each of them is infinite, we ought to find, as we do in fact, that the difference is nothing in comparison with them. When, therefore, it is asked what the result of certain operations on several quantities becomes, on the supposition that x is infinite, it is to the result that we must apply the principle stated above, and not to each of the quantities taken separately. Thus we shall find that the sum of

$$-x^2 + a x + b, \text{ and } x^2 + b x + c,$$

is reduced, when x is infinite, to $a x + b x$; for, by the general rule, it is $a x + b x + b + c$, which, when x is infinite, is reduced to $a x + b x$. In like manner, if we had $x - \sqrt{x^2 - b^2}$; this quantity, when x is infinite, seems to be nothing. But as $\sqrt{x^2 - b^2}$ is only an indication of the root of $x^2 - b^2$, we must, in order to find the difference between this quantity and x, reduce $-\sqrt{x^2 - b^2}$ to a series (*Alg.* 144); the quantity $x - \sqrt{x^2 - b^2}$ will then become $x - x + \frac{b^2}{2x} + \frac{b^4}{8x^3} +$ &c., or $\frac{b^2}{2x} + \frac{b^4}{8x^3} +$ &c., which, when x is infinite with regard to b, is reduced to $\frac{b^2}{2x}$.

Elements of the Differential Calculus.

6. When we consider a variable quantity as increasing by infinitely small degrees, if we wish to know the value of those increments, the mode which most naturally presents itself is, to determine the value of this quantity for any one instant, and the value of the same quantity for the instant immediately following. The difference of these two values is the increment or decrement by which this quantity has been increased or diminished. This difference is also called *the differential* of the quantity.

7. To mark the differential of a simple variable quantity, as x or y, we write $d\,x$ or $d\,y$; that is, we place before the variable the initial d of the word difference. But when we wish to indicate the differential of a compound quantity, as x^2, $5\,x^2+3\,x^2$, or $\sqrt{x^2-a^2}$, &c., we enclose this quantity in a parenthesis, before which we write the letter d; thus we write

$$d\,(x^2),\ d\,(5\,x^3+3\,x^2),\ d\,(\sqrt{x^2-a^2}),\ \&c.$$

The differential of a compound quantity is also sometimes expressed by a point between d and the quantity, as $d\,.\,x^2$, $d\,.\,x\,y\,z$, &c.

We shall hereafter represent the variable quantities by the last letters of the alphabet, t, u, x, y, z; and the constant quantities, or those which always preserve the same value, by the first letters, a, b, c, &c., and if they are used otherwise, notice of it will be given. As to the letter d, it will be used only to designate the differential of the quantity before which it is placed.

8. Agreeably to the idea which has just been given of the differential of a quantity, we see, that to get the differential of a quantity which contains only variables of the first degree, and neither multiplied nor divided by each other, we have only to write the characteristic, d, before each variable, leaving the sign of each unchanged; for example, the differential of $x+y-z$ will be $dx+d\,y-d\,z$. For, in order to obtain this differential, we must consider x as becoming $x+d\,x$; y as becoming $y+dy$; and z as becoming $z+d\,z$; then the quantity proposed, which is $x+y-z$, would become $x+d\,x+y+d\,y-z-d\,z$; and, taking the difference of these two states, we shall have

$$x+d\,x+y+d\,y-z-d\,z-x-y+z;$$

that is,

$$d\,(x+y-z)=d\,x+d\,y-d\,z.$$

The case would be the same, if the variables, which enter into the proposed quantity, had constant coefficients; thus the differential of $5\,x+3\,y$, is $5\,d\,x+3\,d\,y$; that of $a\,x+b\,y$, is $a\,d\,x+b\,d\,y$; for, when x and y become $x+d\,x$ and $y+d\,y$, the quantity $a\,x+b\,y$ becomes $a\,(x+d\,x)+b\,(y+d\,y)$, that is

$$a\,x+a\,d\,x+b\,y+b\,d\,y;$$

the difference of the two states, or the differential, is $a\,d\,x+b\,d\,y$; that is, generally, *each variable must be preceded by the characteristic* d.

If in the proposed quantity there be one term entirely constant, the differential will be the same as if there were no such term. That is, the differential of that term will be nothing. This is evident, since the differential being nothing else but the increment, a constant quantity cannot have a differential without ceasing to be constant; thus the differential of $a\,x+b$ is simply $a\,d\,x$.

9. When the variable quantities are simple but multiplied together we must observe the following rule. *Find the differential of each variable quantity successively, as if all the rest were a constant coefficient.*

For example, to find the differential of $x\,y$, we first consider x as constant and obtain $x\,d\,y$†, we then consider y as constant and have $y\,d\,x$, so that, $d\,(x\,y)=x\,d\,y+y\,d\,x$.

The reason of this rule will be seen by going back to the *principle* upon which it is founded. To find the differential of $x\,y$, we must consider x as becoming $x+d\,x$, that is, as increasing by the infinitely small quantity $d\,x$; and y as becoming $y+d\,y$, that is, increasing by the infinitely small quantity $d\,y$; then $x\,y$ becomes $(x+d\,x)\times(y+d\,y)$, that is, $x\,y+x\,d\,y+y\,d\,x+d\,y\,d\,x$; then the difference of the two states, or the differential, is

$$x\,y+x\,d\,y+y\,d\,x+d\,y\,d\,x-x\,y,$$

or

$$x\,d\,y+y\,d\,x+d\,y\,d\,x;$$

but in order that the calculus may indicate that $d\,y$ and $d\,x$ are infinitely small quantities, as they are supposed to be, we must (5) omit $d\,y\,d\,x$, which (4) is an infinitely small quantity of the second order, and therefore infinitely small compared with $x\,d\,y$ and $y\,d\,x$, which are infinitely small of the first; therefore the differential of $x\,y$, or $d\,.\,x\,y$ is $x\,d\,y+y\,d\,x$, agreeably to the rule.

† To avoid obscurity, in the manner of writing, it is best to write last the variable affected with the characteristic *d*.

We shall find, by the same rule, that the differential of $x\,y\,z$ is $x\,y\,d\,z + x\,z\,d\,y + y\,z\,d\,x$, by differentiating as if $x\,y$, $x\,z$, and $y\,z$ were successively constant quantities. This may be demonstrated, as above, by regarding x, y, and z as becoming, respectively, $x + d\,x$, $y + d\,y$, and $z + d\,z$; in which case $x\,y\,z$ is changed into $(x + d\,x)\,(y + d\,y)\,(z + d\,z) = x\,y\,z + x\,y\,d\,z + x\,z\,d\,y + y\,z\,d\,x + y\,d\,x\,d\,z + z\,d\,y\,d\,x + x\,d\,z\,d\,y + d\,x\,d\,y\,d\,z$; then the difference of the two states will become, by reducing, and rejecting the infinitely small quantities of the second and third orders,

$$d\,.\,x\,y\,z = x\,y\,d\,z + x\,z\,d\,y + y\,z\,d\,x.$$

10. If the quantity proposed be any power of a variable quantity, observe the following rule. *Multiply by the exponent, diminish this exponent by unity, and multiply the result by the differential of the variable.*

Thus, to find the differential of x^2, we first multiply by the exponent 2, diminish this exponent 2 by 1, then multiplying by $d\,x$ the differential of the variable x^2, we have $2\,x\,d\,x$.

We shall find, in the same manner, that the differential x^3 is $3\,x^2\,d\,x$; that of x^4, $4\,x^3\,d\,x$; that of x^{-1}, $-x^{-2}\,d\,x$; that of x^{-3}, $-3\,x^{-4}\,d\,x$; that of $x^{\frac{1}{2}}$, $\frac{1}{2}\,x^{-\frac{1}{2}}\,d\,x$; that of $x^{\frac{4}{3}}$, is $\frac{4}{3}\,x^{\frac{1}{3}}\,d\,x$; and, generally, that of x^m, is $m\,x^{m-1}\,d\,x$; whether the exponent m be positive or negative, a whole number or a fraction.

To find the reason of this rule let us go back to the first principles. Let us consider x as becoming $x + d\,x$ ($d\,x$ being infinitely small); then x^m becomes $(x + d\,x)^m$, which, being reduced to a series (*Alg.* 144), becomes

$$x^m + m\,x^{m-1}\,d\,x + m\,\frac{m-1}{2}\,x^{m-2}\,d\,x^2 + \&c.$$

or, because the term $m\,.\,\frac{m-1}{2}\,.\,x^{m-2}\,d\,x^2$ is infinitely small of the second order, and the following terms would be of still lower orders, the series is reduced to $x^m + m\,x^{m-1}\,d\,x$; then the difference of the two states is $x^m + m\,x^{m-1}\,d\,x - x^m$; and, therefore, $d\,.\,x^m = m\,x^{m-1}\,d\,x$.

If there were a constant coefficient or multiplier, the case would not be altered, the constant coefficient would remain in the differential the same that it is in the quantity;

$$d\,.\,a\,x^m, \text{ therefore, } = m\,a\,x^{m-1}\,d\,x.$$

We have thus given whatever it is necessary to know, in order to be able to differentiate all sorts of algebraical quantities. What follows is only an application of these rules.

11. Suppose it were required to differentiate the fraction $\frac{x}{y}$, we should write it $x y^{-1}$ (*Alg.* 133); and then, applying the rule given (9), we have

$$d \,.\, x y^{-1} = x \, d \,.\, y^{-1} + y^{-1} \, d x,$$

and, consequently, (10) $d \,.\, x y^{-1} = y^{-1} \, d x - x y^{-2} \, d y =$, by reducing to a common denominator, $\frac{y \, d x - x \, d y}{y^2}$.

Therefore, *to find the differential of a fraction, we multiply the differential of the numerator by the denominator, subtract from the product the differential of the denominator, multiplied by the numerator, and divide the whole by the square of the denominator.* This is the rule usually given for the differentiation of fractions; but we easily perceive that we may dispense with charging the memory with this new rule; as it is sufficient to raise the denominator into the numerator (*Alg.* 133), and then differentiate by the general rule.

12. If we wish to differentiate $a x^3 y^2$, we first consider x^3 and y^2 as two simple variables, and (9) we have

$$d \,.\, a x^3 y^2 = a x^3 \, d \,.\, y^2 + a y^2 \, d \,.\, x^3 \,;$$

then (10) we have

$$d \,.\, a x^3 y^2 = 2 a x^3 y \, d y + 3 a y^2 x^2 \, d x.$$

In general,

$$d \,.\, a x^m y^n = a x^m \, d \,.\, y^n + a y^n \, d \,.\, x^m = n a x^m y^{n-1} \, d y + m a y^n x^{m-1} \, d x.$$

13. If the quantity, which we wish to differentiate, is complex, but without containing any powers of complex quantities, we differentiate separately each of the terms of which it is composed. Thus,

$$d (a x^3 + b x^2 + c x y) = 3 a x^2 \, d x + 2 b x \, d x + c x \, d y + c y \, d x.$$

In like manner,

$$d \left(a x^2 + b x + \frac{c y}{x^2}\right) = d \, (a x^2 + b x + c x^{-2} y)$$
$$= 2 a x \, d x + b \, d x - 2 c x^{-3} y \, d x + c x^{-2} \, d y.$$

In like manner,

$$d (x^3 y + a y^2 + b^3) = 3 x^2 y \, d x + x^3 \, d y + 2 a y \, d y,$$

observing that the constant quantity b^3 has no differential.

14. If the exponent of the quantity be a whole number, as in $(a+bx+cx^2)^5$, we regard the whole quantity affected by this exponent as a single variable, and differentiate by the rule for powers (10). Thus,

$$d(a+bx+cx^2)^5 = 5(a+bx+cx^2)^4 \times d(a+bx+cx^2)$$
$$= 5(a+bx+cx^2)^4 \times (b\,dx \times 2cx\,dx).$$

In like manner,

$$d(a+bx^2)^{\frac{5}{3}} = \tfrac{5}{3}(a+bx^2)^{\frac{2}{3}} \times d(a+bx^2)$$
$$= \tfrac{5}{3}(a+bx^2)^{\frac{2}{3}} \times 2bx\,dx = \tfrac{10}{3}\,b\,x\,dx\,(a+bx^2)^{\frac{2}{3}}.$$
$$d(x^2+2ax+a^2)^2 = 4(x^2+2ax+a^2)(x+a)\,dx.$$

15. When a complex quantity is composed of different factors, we regard each factor as a simple variable, and follow the rule given (9) for a product of several simple variables. Thus $x^3(a+bx^2)^{\frac{5}{3}}$, which we may consider as composed of the two factors x^3 and $(a+bx^2)^{\frac{5}{3}}$, will give

$$d(x^3(a+bx^2)^{\frac{5}{3}}) = (a+bx^2)^{\frac{5}{3}}\,d(x^3) + x^3\,d(a+bx^2)^{\frac{5}{3}},$$

which, by the preceding rules, becomes

$$3x^2\,dx\,(a+bx^2)^{\frac{5}{3}} + \tfrac{10}{3}\,b x^4\,dx\,(a+bx^2)^{\frac{2}{3}}.$$

And

$$d\left(\frac{(x+a)^3}{(x+b)^2}\right) = d\{(x+a)^3(x+b)^{-2}\} = (x+a)^3\,d(x+b)^{-2}$$
$$+ (x+b)^{-2}\,d(x+a)^3;$$

that is,

$$= -2(x+a)^3(x+b)^{-3}\,dx + 3(x+b)^{-2}(x+a)^2\,dx,$$

which, by restoring the denominators, becomes

$$-\frac{2(x+a)^3\,dx}{(x+b)^3} + \frac{3(x+a)^2\,dx}{(x+b)^2}, =$$

(reducing to a common denominator)

$$-\frac{2(x+a)^2(x+a)\,dx}{(x+b)^3} + \frac{3(x+a)^2(x+b)\,dx}{(x+b)^3}$$
$$= \frac{-2(x+a)^2(x+a)\,dx + 3(x+a)^2(x+b)\,dx}{(x+b)^3}$$
$$= \frac{(3x+3b-2x-2a)(x+a)^2\,dx}{(x+b)^3} = \frac{(x+3b-2a)(x+a)^2\,dx}{(x+b)^3}.$$

Also, $$d\left(\frac{a}{(x^2+y^2)^{\frac{1}{2}}}\right) = -\frac{a(x\,dx+y\,dy)}{(x^2+y^2)^{\frac{3}{2}}}.$$

16. If the proposed quantity is radical, we substitute fractional exponents in place of the radical signs (*Alg.* 132), and differentiate according to the rules already given. Thus,

$$d(\sqrt{x}) = d(x^{\frac{1}{2}}) = \tfrac{1}{2}x^{-\frac{1}{2}}\,dx = \frac{dx}{2\sqrt{x}};\dagger$$

$$d(\sqrt[5]{x^3}) = d(x^{\frac{3}{5}}) = \tfrac{3}{5}x^{-\frac{2}{5}}\,dx;$$

$$d(\sqrt{a^2-x^2}) = d(a^2-x^2)^{\frac{1}{2}} = \tfrac{1}{2}(a^2-x^2)^{-\frac{1}{2}}d(a^2-x^2) =$$
$$-x\,dx(a^2-x^2)^{-\frac{1}{2}} = \frac{-x\,dx}{\sqrt{a^2-x^2}};$$

$$d\left\{x^m\sqrt[q]{(a+bx^n)^p}\right\} = d\left\{x^m(a+bx^n)^{\frac{p}{q}}\right\}$$
$$= x^m\,d(a+bx^n)^{\frac{p}{q}} + (a+bx^n)^{\frac{p}{q}}dx^m$$
$$= \frac{bnp}{q}x^{m+n-1}\,dx \times (a+bx^n)^{\frac{p}{q}-1} + m\,x^{m-1}\,dx\,(a+bx^n)^{\frac{p}{q}}.$$

In like manner,

$$d.(x+y^{\frac{1}{2}})^{\frac{1}{2}} = \tfrac{1}{2}(x+y^{\frac{1}{2}})^{-\frac{1}{2}}d(x+y^{\frac{1}{2}}) = \frac{\frac{1}{2}y^{\frac{1}{2}}dx + \frac{1}{4}dy}{(xy+y^{\frac{3}{2}})^{\frac{1}{2}}}.$$

$$d\left(\frac{a}{(a-x)^3}\right) = \frac{3a\,dx}{(a-x)^4}. \quad d\left(\frac{x}{1+x}\right) = \frac{dx}{(1+x)^2}$$

$$d\left(\frac{x^n}{(1+x)^n}\right) = \frac{nx^{n-1}dx}{(1+x)^{n+1}}.$$

Of Second, Third, &c. Differentials.

17. In addition to the differentials which we have just been considering, and which are called *first differentials*, we consider also second, third, &c. differentials. These are indicated by writing twice the characteristic, d, before the variable, for the second differentials, three times for the third, &c. For example, $d\,d\,x$ indicates the second differential of x; $d\,d\,d\,x$, the third differential.

When we speak of second differentials, we consider the variable as increasing by increments which are unequal, but whose differential is infinitely small with regard to these increments themselves. Thus $d\,d\,x$ is infinitely small compared with dx. In the third differentials also, $d\,d\,d\,x$ or $d^3\,x$, (for they are indicated in both ways), is infinitely small compared with $d\,d\,x$, and so on. To indicate the square of $d\,x$,

† From this expression we may deduce the rule, *to differentiate a radical of the second degree, we divide the differential of the quantity under the radical sign, by double the radical itself.*

we should naturally write $(dx)^2$; but, for greater simplicity, we write dx^2, which cannot be mistaken for the differential of x^2, as that is designated by $d\,(x^2)$ or $d\,.\,x^2$.

We observe that although ddx and dx^2 are both infinitesimals of the second order, they are nevertheless not equal; for ddx is the second differential of x, or the difference of two successive differentials of x; and dx^2 is the square of dx.

In order to determine the second differentials, it is most natural to consider the variable quantity in three successive states, infinitely near each other; to take the difference between the second state and the first, that between the third and second, and then take the difference of these two differences. For example, the first state of x is x; at the second instant it has increased by the quantity dx, and become $x + dx$; the following instant $x + dx$ increases by $dx + d\,(dx)$, $d\,(dx)$ marking the quantity by which the increment of the second instant exceeds that of the first, or the differential of dx. Thus the three successive states of the quantity x are

$$x, x + dx, x + 2\,dx + d\,(dx).$$

The difference between the second and first is dx; that between the third and second is $dx + d\,(dx)$; finally, the difference between these two differentials, or the second differential of x, is $d\,(dx)$; we have therefore $ddx = d\,(dx)$. Therefore, *the second differentials are obtained by differentiating the first differentials according to the rules already given.*

For example, to get the second differential of xy, we take the first differential, which is $x\,dy + y\,dx$; we then differentiate this quantity as if x and dx, y and dy, were so many different variables, and we find $$x\,ddy + dy\,dx + y\,ddx + dy\,dx;$$
or $$dd.\,xy = x\,ddy + 2\,dy\,dx + y\,ddx.$$

In like manner the second differential of x^2 is found by first differentiating x^2, which gives $2\,x\,dx$; then differentiating $2\,x\,dx$ as if x and dx were both finite variables, which gives

$$2\,x\,ddx + 2\,dx^2.$$

We shall find also, that

$$dd.\,ax^m = d\,.\,max^{m-1}dx = m\,.\,m - 1\,ax^{m-2}dx^2 + max^{m-1}ddx.\dagger$$

† A difficulty may present itself in this mode of taking the second differentials which it is best to explain. When we determine the first differentials, we reject the infinitely small quantities of the second order; but the second differentials being infinitely small of the second order, is there not reason to fear that what we have rejected in the valuation of the first, will render the second defective? We answer, no: for this infinitely small quantity of the second order, which has been rejected, can have for its differential only an infinitely small quantity of the

radius is unity, is found by multiplying the differential of the angle by the cosine of the same angle.

22. In like manner, the differential of $\cos z$, or

$\cos(z + dz) - \cos z = \cos z \cos dz - \sin z \sin dz - \cos z$,

since (*Trig.* 11)

$$\cos(z + dz) = \cos z \cos dz - \sin z \sin dz;$$

therefore, as

$$\sin dz = dz, \text{ and } \cos dz = 1,$$

we obtain

$$d(\cos z) = \cos z - dz \sin z - \cos z = -dz \sin z;$$

that is to say, *The differential of the cosine of an angle, whose radius is* 1, *is found by multiplying the differential of the angle taken with the contrary sign, by the sine of the same angle.*

Thus, to recapitulate, we have

$$d(\sin z) = dz \cos z; \quad d(\cos z) = -dz \sin z.$$

By means of these two principles, we may differentiate any quantity composed of sines and cosines, without any other rules than those already given.

Thus, to differentiate $\cos 3z$, we have

$$d(\cos 3z) = -3dz \sin 3z.$$

Universally, if m is a constant quantity,

$$d(\cos mz) = -m dz \sin mz; \quad d(\sin mz) = m dz \cos mz.$$

In like manner

$$d(\sin z \cos t) = \cos t\, d(\sin z) + \sin z\, d(\cos t)$$
$$= dz \cos t \cos z - dt \sin z \sin t.$$

And

$$d(\sin z)^m = m(\sin z)^{m-1} d(\sin z) = m dz \cos z (\sin z)^{m-1}.$$

23. If we had $\frac{\sin z}{\cos z}$, which is the expression for the tangent of an angle when radius $= 1$, since (*Trig.* 8)

$$\cos z : 1 :: \sin z : \text{tang } z,$$

we should have

$$d\left(\frac{\sin z}{\cos z}\right) = d.\sin z(\cos z)^{-1} = dz \cos z(\cos z)^{-1} + dz \sin z^2 \cos z^{-2}$$

$$= \frac{dz \cos z}{\cos z} + \frac{dz \sin z^2}{\cos z^2} = \frac{dz \cos z^2 + dz \sin z^2}{\cos z^2} = \frac{dz}{\cos z^2},$$

because (*Trig.* 10) $\cos z^2 + \sin z^2 = 1$. Therefore,

The differential of the tangent of an angle, whose radius is 1, *is equal to the differential of the angle, divided by the square of the cosine of the same angle.*

Whence we may also conclude, that *the differential of an angle is equal to the differential of the tangent of that angle, multiplied by the square of its cosine;* for, since

$$d\left(\frac{\sin z}{\cos z}\right) = d\ (\text{tang}\ z) = \frac{d\,z}{\cos z^2},$$

we have $$d\,z = \cos z^2\, d\ \text{tang}\ z.$$

24. If it were required to differentiate $\frac{\cos z}{\sin z}$, which is the expression for the cotangent of the angle z, we should have

$$d\,.\frac{\cos z}{\sin z} = d\,.\cos z \sin z^{-1} = -d\,z \sin z \sin.z^{-1} - dz \cos z^2 \sin z^{-2}$$

$$= -\frac{dz \sin z}{\sin z} - \frac{d\,z \cos z^2}{\sin z^2} = \frac{-d\,z \sin z^2 - d\,z \cos z^2}{\sin z^2} = \frac{-d\,z}{\sin z^2}.$$

Therefore, *The differential of the cotangent of an angle, is equal to the differential of the angle, taken negatively, divided by the square of the sine of the same angle.* The use of these differentiations will be exemplified hereafter.

Of Logarithmic Differentials.

25. According to the description already given (*Alg.* 238), logarithms are a series of numbers in any arithmetical progression, answering, term by term, to a series of numbers in any geometrical progression.

This being laid down, let y and y' be two consecutive terms of a geometrical progression, of which r is the ratio, and a and a' the two first terms. Let, also, x and x' be two consecutive terms of an arithmetical progression, of which b and b' are the two first terms. Let us suppose, moreover, that x and x' are in the same place in the arithmetical progression that y and y' are in the geometrical progression; in which case, x and x' are the logarithms of y and y'.

By the nature of geometrical progression (*Alg.* 231), we have $y' = r\,y$, and $a' = r\,a$; substituting in the first of these equations the value of r deduced from the second, we have

$$y' = \frac{a'\,y}{a} \quad \text{or} \quad \frac{y'}{y} = \frac{a'}{a}.$$

Let us now suppose that the difference between y' and y is z, or

that $y' = y + z$; we shall have $\frac{y+z}{y}$ or $1 + \frac{z}{y} = \frac{a'}{a}$, and consequently, $\frac{z}{y} = \frac{a'}{a} - 1 = \frac{a' - a}{a}$, or $\frac{a\,z}{y} = a' - a$.

Again, the nature of arithmetical progression gives (*Alg.* 228) $x' - x = b' - b$.

In order to find then the ratio of these two progressions, let us suppose that the difference $a' - a$ of the two first terms of the geometrical progression, is to the difference $b' - b$ of the two first terms of the arithmetical progression as unity is to any number m; that is to say, that $a' - a : b' - b :: 1 : m$; we shall have $m\,(a' - a) = b' - b$; substituting then, in this last equation, instead of $a' - a$ and $b' - b$, the values which have just been found, we shall have $\frac{m\,a\,z}{y} = x' - x$, an equation which expresses generally the ratio of any geometrical progression to any corresponding arithmetical progression.

Let us imagine that, in each of these progressions, the consecutive terms are infinitely near each other; then z, which marks the difference of y' and y, will be $d\,y$; and $x' - x$, which marks the difference of x' and x, will be $d\,x$; whence, the equation will be changed into $\frac{m\,a\,d\,y}{y} = d\,x$.

With regard to m, which indicates the ratio of the difference of the first two terms of the arithmetical progression, to the difference of the first two terms of the geometrical progression, it will nevertheless be a finite number, although these two differences be infinitely small, because we easily conceive that one of two infinitely small quantities may contain the other as many times as one of two finite quantities can contain the other.

The equation $\frac{m\,a\,d\,y}{y} = d\,x$ shows, therefore, that $d\,x$, the differential of the logarithm of a number represented by y, is equal to $d\,y$, the differential of that number, divided by the same number y, and multiplied by the first term a of the fundamental geometrical progression, and by the number m, which represents the ratio of the difference of the first two terms of the arithmetical progression to the difference of the first two terms of the geometrical progression. As this number, m, determines, in some measure, the relation of the two progressions, it is called the *modulus.*

We see then, that according to the value which m and the first term a of the geometrical progression are supposed to have, the same number y may have different logarithms. But of all these different systems of logarithms, the most convenient in algebraical calculations, is that in which the first term of the geometrical progression is 1, and in which the modulus is 1. In that case, the equation $\frac{m\,a\,d\,y}{y} = d\,x$, which comprehends all the different systems of logarithms, becomes

$$\frac{d\,y}{y} = d\,x.$$

26. In the system of logarithms, therefore, used in algebraical calculations, *the differential* d x *of the logarithm* x *of any number* y, *is equal to* d y, *the differential of that number, divided by the number itself.* This is the principle by which we may easily find the differential of the logarithm of any algebraical quantity. But before making use of it, we must observe,

1st. That the logarithms here spoken of are not those of the tables; but it is easy to deduce the one from the other, as will be seen hereafter.

2d. That since the first term b of the arithmetical progression is not found in the equation $\frac{m\,a\,d\,y}{y} = d\,x$, this equation, as well as the particular equation $\frac{d\,y}{y} = d\,x$ just deduced from it, are always true, whatever may be the first term b, that is to say, the logarithm of the first term a of the geometrical progression. We are therefore at liberty to suppose, for the sake of greater simplicity, that the first term of the arithmetical progression is nothing; and, as the geometrical progression which has been fixed upon has unity for its first term, we shall take zero or 0 for the logarithm of 1; but it should be observed that this is entirely arbitrary.

By thus taking unity for the first term of the geometrical progression, and zero for the first term of the arithmetical progression, or for the logarithm of unity, the rules already given (*Alg.* 241) for the application of logarithms, will equally well apply here. If we generalize these rules, designating logarithm by l, we shall see that, instead of $l\ (a\ b)$ we may take $l\ a + l\ b$; instead

of $l\,\frac{a}{b}$, $l\,a - l\,b$. In the same manner, $l\,a^m = m\,l\,a$; finally, $l\sqrt[n]{a^m} = l\,a^{\frac{m}{n}} = \frac{m}{n}\,l\,a$.

This being laid down, if we apply the principle which has just been established concerning the differential of the logarithm of a number, we shall find that

$$d\,l\,x = \frac{d\,x}{x};\ d\,l\,(a+x) = \frac{d\,(a+x)}{a+x} = \frac{d\,x}{a+x};$$

$$d\,l\left(\frac{a}{a+x}\right) = d\,(l\,a - l\,(a+x)) = -\frac{d\,(a+x)}{a+x} = -\frac{d\,x}{a+x},$$

observing that the differential of the constant, $l\,a = 0$.

We have also,

$$d\,.\,l\frac{1}{x} = d(l\,1 - l\,x) = -\frac{d\,x}{x};\ d\,.\,l\,x^2 = d\,.\,2\,l\,x = \frac{2\,d\,x}{x};$$

$$d\,.\,l\,(x\,y) = d\,(l\,x + l\,y) = \frac{d\,x}{x} + \frac{d\,y}{y};$$

$$d\,l\,\frac{x}{y} = d\,(l\,x - l\,y) = \frac{d\,x}{x} - \frac{d\,y}{y};$$

$$d\,l\,\left(\frac{a+x}{a-x}\right) = d\left(l\,(a+x) - l\,(a-x)\right) = \frac{d\,x}{a+x} + \frac{d\,x}{a-x};$$

$$d\,l\,(a^2 + x^2) = \frac{d\,(a^2+x^2)}{a^2+x^2} = \frac{2\,x\,d\,x}{a^2+x^2};$$

$$d\,l\sqrt{a^2+x^2} = \frac{d\sqrt{a^2+x^2}}{\sqrt{a^2+x^2}} = \frac{x\,d\,x}{\sqrt{a^2+x^2}\sqrt{a^2+x^2}} = \frac{x\,d\,x}{a^2+x^2};$$

or, more directly,

$$d\,l\sqrt{a^2+x^2} = d\,.\,\tfrac{1}{2}\,l\,(a\ + x^2) = \frac{x\,d\,x}{a^2+x^2};$$

$$d\,l\,\left(x^m(a+b\,x^n)^p\right) = d\,\left(l\,x^m + l\,(a+b\,x^n)^p\right)$$

$$= d\left(m\,l\,x + p\,l(a+b\,x^n)\right) = \frac{m\,d\,x}{x} + \frac{n\,p\,b\,x^{n-1}d\,x}{a+b\,x^n}.$$

These examples are sufficient to show how other logarithmic quantities may be differentiated.

Of the Differentials of Exponential Quantities.

27. We sometimes meet with quantities of this form, c^x, x^y; that is to say, quantities whose exponent is variable. They are called *Exponential Quantities.*

In order to find how to differentiate these quantities, let us suppose

$$x^y = z;$$

then, taking the logarithms of each member, we have

$$l\,x^y = l\,z$$

and consequently $$d\,l\,(x^y) = \frac{d\,z}{z};$$

then $$d\,z = z\,d\,l\,(x^y),$$

and, substituting for z and $d\,z$, their values,

$$d\,(x^y) = x^y\,d\,l\,(x^y);$$

that is to say, *the differential of an exponential quantity is found by multiplying that exponential quantity by the differential of its logarithm.* Thus,

$$d\,.\,x^y = x^y\,d\,.\,l\,x^y = x^y\,d\,.\,y\,l\,x = x^y\left(d\,y\,l\,x + \frac{y\,d\,x}{x}\right).$$

In like manner,

$$d\,(a^x + y^z) = d\,.\,a^x + d\,.\,y^z = a^x\,d\,.\,l\,a^x + y^z\,d\,.\,l\,y^z$$

$$= a^x\,d\,.\,x\,l\,a + y^z\,d\,.\,z\,l\,y = a^x\,d\,x\,l\,a + y^z\left(d\,z\,l\,y + \frac{z\,d\,y}{y}\right).$$

So

$$d\,(a^2+x^2)^x = (a^2+x^2)^x\,d\,l\,(a^2+x^2)^x = (a^2+x^2)^x\,d\,.\,x\,l\,(a^2+x^2)$$

$$= (a^2+x^2)^x\left(d\,x\,l\,(a^2+x^2) + \frac{2\,x^2\,d\,x}{a^2+x^2}\right);$$

and so of others.

Frequent use is made, in calculations, of the exponential quantity e^x, e being the number whose logarithm is $= 1$. The differential of this quantity is, according to what has just been laid down,

$$e^x\,d\,.\,l\,e^x = e^x\,d\,(x\,l\,e) = e^x\,d\,x\,l\,e;$$

and since $l\,e$ is supposed $= 1$, we have simply

$$d\,.\,e^x = e^x\,d\,x.$$

That is to say, *this particular exponential has, for its differential, the exponential itself multiplied by the differential of its exponent.* This exponential will be found hereafter.

Application of the preceding Rules.

28. In order to show, by some examples, the use of the rules which have just been given, and the advantage which they have over common algebra, we shall now apply them to subjects with which we are already acquainted, viz. questions in Geometry and Algebra.

Application to the Subtangents, Tangents, Subnormals, &c. of Curved Lines.

29. To draw a tangent to any curved line AM (*fig.* 1), we conceive this curve to be a polygon of an infinite number of infinitely small sides. The prolongation MT of Mm, one of these sides, is the tangent, which is determined for each point M, by calculating the value of the subtangent PT, or that part of the line of abscissas, which is comprehended between the ordinate PM and the point T, the intersection of this tangent. The subtangent is determined in the following manner.

Through the extremities M and m of the infinitely small side Mm, we suppose the two ordinates MP and mp, to be drawn, and through the point M, the line Mr parallel to AP, the axis of the abscissas. The infinitely small triangle Mrm is similar to the finite triangle TPM, and gives this proportion,

$$rm : rM : : PM : PT.$$

Now if we call AP, x; PM, y; it is evident that Pp, or its equal rM, will be dx, and rm will be dy; we shall therefore have

$$dy : dx : : y : PT = \frac{y\,dx}{dy}.$$

This is the general formula for determining the subtangent of any curve whatever, whether the y's and x's are perpendicular to each other or not, provided that the y's are parallel among themselves. We shall now give an example of the application of this formula to any curve of which we have the equation.

Let us suppose that the nature of any curve AM were expressed by an equation containing x, y, and constant quantities. If we differentiate this equation, there can never be more than two kinds of terms, those multiplied by dx and those by dy. It will then be easy, by the common rules of Algebra, to deduce, from

this differential equation, a value for $\frac{dx}{dy}$, which shall contain only terms of x, y, and constants; by substituting this value in the formula $\frac{y\,dx}{dy}$ or $y \times \frac{dx}{dy}$, we shall have a value for the subtangent in x, y, and constants; finally, putting instead of y its value in terms of x, deduced from the equation of the curve, we shall have the value of the subtangent expressed in terms of x and constant quantities only. So that to determine the position of this line for any point whatever M, we have only to substitute, in this last result, in place of x, the value of the abscissa AP, which answers to that point.

Suppose, for example, that the given curve is an ellipse, of which the equation is (*Ap.* 112) $y^2 = \frac{b^2}{a^2}(a\,x - x^2)$. Differentiating this equation, we have

$$2y\,dy = \frac{b^2}{a^2}(a\,dx - 2\,x\,dx) \text{ or } 2\,a^2\,y\,dy = a\,b^2\,dx - 2\,b^2\,x\,dx;$$

from this we deduce the value of $\frac{dx}{dy}$, by dividing first by $d\,y$ and then by the multiplier of $d\,x$, in the second member, and find $\frac{dx}{dy} = \frac{2\,a^2\,y}{ab^2 - 2\,b^2\,x}$; substituting this value in $\frac{y\,dx}{dy}$, we shall have

$$\frac{y\,dx}{dy} = \frac{2\,a^2\,y^2}{a\,b^2 - 2\,b^2\,x};$$

finally, substituting for y^2, its value $\frac{b^2}{a^2}(a\,x - x^2)$ given by the equation of the curve, and reducing, we have

$$\frac{y\,dx}{dy} = PT = \frac{2\,(a\,x - x^2)}{a - 2\,x} = \frac{a\,x - x^2}{\frac{1}{2}a - x},$$

a value which is precisely the same as that which was found by Algebra (*Ap.* 119), but which is obtained here in a more expeditious manner.

We may observe here how this result justifies what was said (5) concerning the quantities which are to be rejected in the culculation; for, by employing here the differential calculus, the rules of which, in this example, suppose the omission of the infinitely small quantities of the second order, by the side of those of the first, we arrive at the same result as in the Application

of Algebra to Geometry, where this subtangent was determined in the most exact and rigorous manner. We see that by thus rejecting the quantities which were pointed out to be neglected, we only impress upon the calculus the character which it ought to have, in order to express the conditions of the question.

We pursue a similar course in determining the tangents, subnormals, normals, &c.

Let us suppose, for the sake of greater simplicity, that the abscissas and ordinates, (the x's and y's), are perpendicular to each other. In order to determine the tangent, we compare anew the triangle $Mm\,r$ with the triangle TPM, and we have

$$rm : Mm :: PM : TM;$$

but by a property of the right angled triangle, $Mr\,m$, we have

$$Mm = \sqrt{\overline{rM}^2 + \overline{rm}^2} = \sqrt{dx^2 + dy^2}\,;$$

therefore, $$dy : \sqrt{dx^2+dy^2} :: y : TM;$$

therefore, $$TM = \frac{y\sqrt{dx^2+dy^2}}{dy} = \frac{y\sqrt{dx^2+dy^2}}{\sqrt{dy^2}}$$
$$= y\sqrt{\frac{dx^2+dy^2}{dy^2}} = y\sqrt{\frac{dx^2}{dy^2}+1}.$$

Thus, after differentiating the equation of the curve, we shall deduce thence the value of $\frac{dx}{dy}$, the square of which we substitute in this expression for the tangent; after which, putting in place of y its value, in terms of x and constants, drawn from the same equation, we shall have the tangent expressed in terms of x and constant quantities. An application of this may be made to the equation of the ellipse, and the same value will be found as was formerly obtained (*Ap.* 121).

If the subnormal is required, we suppose the line MQ perpendicular to the tangent TM, and observing that the triangles $Mr\,m$, MPQ, which have their sides respectively perpendicular each to each, are similar; we have

$$Mr : rm :: PM : PQ;$$

that is to say,

$$dx : dy :: y : PQ = \frac{y\,dy}{dx}.$$

Then, after having differentiated the equation of the curve, we deduce thence the value of $\frac{dy}{dx}$, which we substitute in $\frac{y\,dy}{dx}$, and,

completing the operation as before, we have the value of the subnormal in terms of x and constant quantities.

In the ellipse, for example, the equation $y^2 = \frac{b^2}{a^2}(a x - x^2)$, being differentiated, gives

$$2 y\, d y = \frac{b^2}{a^2}(a\, d x - 2 x\, d x)\,; \text{ then } \frac{d y}{d x} = \frac{\frac{b^2}{a^2}(a - 2 x)}{2 x}\,;$$

consequently the subnormal,

$$\frac{y\, d y}{d x} = \frac{b^2}{a^2} \times \frac{a - 2 x}{2} = \frac{b^2}{a^2}(\tfrac{1}{2} a - x),$$

as was formerly found (*Ap.* 118).

If the normal MQ were required, it might be found by comparing anew the triangle $M r m$ with the triangle MPQ.

Let us take, as a second example of the formula for subtangents and of that for subnormals, the equation of the parabola, which is $y^2 = p x$ (*Ap.* 172). By differentiating, we have

$$2 y\, d y = p\, d x\,; \text{ then } \frac{d x}{d y} = \frac{2 y}{p}, \text{ and } \frac{d y}{d x} = \frac{p}{2 y};$$

the subtangent, therefore,

$$\frac{y\, d x}{d y} = \frac{2 y^2}{p} = \frac{2 p x}{p} = 2 x,$$

and the subnormal

$$\frac{y\, d y}{d x} = \frac{p y}{2 y} = \frac{p}{2},$$

which agree perfectly with what have been already found (*Ap.* 179, 180).

We shall take, as a third example, the equation $y^{m+n} = a^m x^n$, which is a general expression for parabolas of every kind.

The name of parabola is given to every curve, whose equation, such as $y^{m+n} = a^m x^n$, has only two terms, but in which, the exponents of x and y, in the two members, have the same sign.

By differentiating this equation, we have

$$(m + n)\, y^{m+n-1} d y = n\, a^m x^{n-1} d x\,;$$

from which $$\frac{d x}{d y} = \frac{(m + n)\, y^{m+n-1}}{n\, a^m x^{n-1}}$$

and the subtangent

$$\frac{y\, d x}{d y} = \frac{(m + n)\, y^{m+n}}{n\, a^m x^{n-1}}\,;$$

and substituting for y^{m+n} its value $a^m x^n$, we find

$$\frac{y\,d\,x}{d\,y}=\frac{(m+n)\,a^m\,x^n}{n\,a^m x^{n-1}}=\frac{m+n}{n}x.$$

Whence we may see that the subtangent, in these curves, is equal to as many times the abscissa x, as there are units in the exponent of y divided by the exponent of x. This holds true, as we have already seen, in the common parabola, where the subtangent is $2\,x$, and where the exponent of y divided by the exponent of x, is in fact $\frac{2}{1}$ or 2.

Let us now take, as an example, a curve the nature of which is expressed by an equation in terms of the differentials of the coördinates. Let us suppose, for instance, the curve *BM* (*fig.* 4) to be such, that the abscissas *AP*, *Ap*, &c. being taken in arithmetical progression, the corresponding ordinates *PM*, *p m*, &c. are in geometrical progression. This is called the *logarithmic curve*, because, while the ordinates represent successively all imaginable numbers, the abscissas are their logarithms, and it will have for its equation $\frac{a\,m\,d\,y}{y}=d\,x$, since we have found (25) that this equation expresses the relation of numbers to their logarithms. We shall have, therefore, $\frac{d\,x}{d\,y}=\frac{a\,m}{y}$, and consequently the subtangent $\frac{y\,d\,x}{d\,y}$ will become $\frac{a\,m\,y}{y}=a\,m$; that is to say, for each point of the same logarithmic, the subtangent PT is always the same, and is equal to as many times the first ordinate AB, or a, as there are units in the modulus m.

30. When the equation of the curve is such that, x increasing, y diminishes, as in figure 2, then the line $r\,m$ must be expressed by $-\,d\,y$ (20); and the proportion $r\,M : r\,m :: PM : PT$, which serves to find the subtangent, becomes

$$-\,d\,y : d\,x :: y : PT = -\frac{y\,d\,x}{d\,y}.$$

Thus there will be no difference in the calculation; the only change is that the tangent, instead of falling on one side of the point A, the origin of the abscissas, with regard to the ordinate PM, will fall on the opposite side. This is the reason why we may always take $\frac{y\,dx}{d\,y}$ as the formula of the subtangents; if the ordinates decrease, the value of $\frac{y\,d\,x}{d\,y}$ will have a negative sign,

which indicates that this value must be referred to the side opposite to the origin of the abscissas.

If, for example, we take the equation of the circle, the origin of the abscissas being at the centre, that is to say, the equation (*Ap.* 103) being $y^2 = \frac{1}{4} a^2 - x^2$, it is evident that, while Cp or x (*fig.* 3) increases, y or PM diminishes; so that the subtangent PT falls on the side of PM oposite to C, the origin of the abscissas. This is shown also by the calculus; for, if we differentiate, we have $2\,y\,d\,y = -2\,x\,d\,x$, and consequently $\frac{d\,x}{d\,y} = \frac{-y}{x}$; then $\frac{y\,d\,x}{d\,y} = \frac{-y^2}{x} = \frac{-(\frac{1}{4}\,a^2 - x^2)}{x}$, a value, of which the sign $-$ indicates, that it should be referred to the side opposite to that which is supposed in taking $\frac{y\,d\,x}{d\,y}$ for the formula of the subtangent.

Let us now take, as an example, the equation $x\,y = a^2$, which belongs to the hyperbola considered between its asymptotes (*Ap.* 163), we have $y\,d\,x + x\,d\,y = 0$, and consequently,

$$\frac{d\,x}{d\,y} = \frac{-x}{y}\,;\ \text{therefore,}\ \frac{y\,d\,x}{d\,y} = \frac{-x\,y}{y} = -x\,;$$

which shows that, to draw a tangent to the hyperbola considered between its asymptotes, we must take upon the asymptote nearest M, the point in question (*fig.* 7), and on the side of PM, which is opposite to A, the origin of the abscissas, the line

$$PT = AP = x.$$

We see with what facility all these results are obtained by the differential calculus.

In the same manner with parabolas of all kinds, we call by the name of *Hyperbolas referred to their asymptotes*, all those curves whose equation, such as $y^m = a^{m+n} x^{-n}$, contains only two terms, but in which the exponents of y and x, in the two members, have contrary signs. We leave these curves as an example for the exercise of the reader. The subtangent will be found to be $= -\frac{m}{n}\,x$; that is to say, that it falls on the side opposite to the origin of the abscissas, and that it is equal to as many times the abscissa as there are units in the exponent of y, divided by the exponent of x.

In general, we determine at once, by this calculus, all the subtangents, tangents, &c. of all the curves of the same family. Those are said to be of the same family, whose equation is formed in the same manner, and differs only in the magnitude of the exponent. Thus we call by the common name of *circles*, all the curves in which any power of the ordinate is equal to the product of any two powers of the two distances of that ordinate from the extremity of the line a, on which the abscissas are taken. Their equation is $y^{m+n} = x^m (a - x)^n$, which comprehends the circle properly so called, when $m = n = 1$. The equation

$$y^{m+n} = \frac{c}{b} x^m (a - x)^n$$

represents ellipses of all kinds; and $y^{m+n} = \frac{c}{b} x^m (a + x)^n$, as well as $y^{m+n} = \frac{c}{b} (x - a)^n$ belongs to the family of hyperbolas. The figure of these curves is determined by means of their equation, as we have seen in conic sections (*Ap.* 101); observing that y has only one real value when its exponent is odd, and two when it is even (*Alg.* 157); which determines that to the same abscissa, there corresponds only one branch of the curve in the first case, and in the second, two, which fall on different sides of the same axis.

31. When we know how to determine the tangents, normals, &c. we may easily solve the two following problems. 1st. *From a given point without a curve, to draw a tangent to this curve.* 2d. *From a point given any where in the plane of a curve line, to draw a perpendicular to that curve.*

For the first of these, let us suppose that DM (*fig.* 8) is the tangent required, which passes through the point D. The general value of PT or $\frac{y\,dx}{dy}$ will be easily found by means of the equation of the curve. If BD be drawn parallel to the ordinate PM, the line BD and its distance BA from A, the origin of the abscissas, are considered as known, since the point D is given in position. Calling, then, DB, h; AB, g; AP, x; and PM, y; we shall have

$$BP = g + x, \text{ and } TB = PT - BP = \frac{y\,dx}{dy} - x - g.$$

Now the similar triangles TBD, TPM, give

$$TB : BD :: TP : PM.$$

That is to say,

$$\frac{y\,dx}{dy} - x - g : h :: \frac{y\,dx}{dy} : y, \text{ or } :: \frac{dx}{dy} : 1$$

Therefore, $$\frac{y\,dx}{dy} - x - g = \frac{h\,dx}{dy}.$$

Putting, therefore, in this equation, the value of $\frac{dx}{dy}$, drawn from the equation of the curve which has been differentiated, and then substituting for y its value in terms of x and constants, also given by the equation of the curve, and we shall have the abscissa x of the point of contact. And if there should be more tangents than one, drawn through the point D, the last equation, in terms of x, will give all the abscissas of the points in which these tangents must end.

As to the case of the perpendicular, let us suppose DQ (*fig.* 5) to be the perpendicular required; the subnormal PQ will be $\frac{y\,dy}{dx}$ (29); now the similar triangles DBQ and MPQ give

$$DB : BQ :: PM : PQ,$$

that is to say, employing the same denominations as above,

$$h : g + x + \frac{y\,dy}{dx} :: y : \frac{y\,dy}{dx}, \text{ or } :: 1 : \frac{dy}{dx}.$$

therefore,

$$\frac{h\,dy}{dx} = g + x + \frac{y\,dy}{dx},$$

an equation of which the same use may be made as in the preceding case.

The two solutions just given may be simplified by making the axis of the abscissas pass through the given point D, in a direction parallel to its former position; that is to say, by taking DK (*fig.* 8) for the axis of the abscissas instead of TP, which only requires, if we make $KM = z$, and consequently, $z = y - h$, or $y = z + h$, that we substitute, both in the equation of the curve and in that of the problem, $z + h$ instead of y. We may also take the point D for the origin of the abscissas.

When the curve has a centre, as the circle, the ellipse, &c. we may always suppose that the point D is in a diameter, and then the solution becomes much more simple.

32. We may remark here, that $\frac{dx}{dy}$ expresses the tangent of the angle which the curve makes, at each point, with the ordinate; and $\frac{dy}{dx}$, the value of the angle made by the element of the curve with the axis of the abscissas. For, in the right angled triangle $M\,r\,m$ (*fig.* 1), we have, supposing the radius of the tables $= 1$, $\quad r\,m : r\,M :: 1 : \text{tang}\, r\,m\,M.$

Therefore $\quad \text{tang}\, r\,m\,M = \frac{r\,M}{r\,m} = \frac{dx}{dy}.$

If then we wish to know in what place, a curve, or its tangent, makes with the ordinate a given angle, or one whose tangent is known, representing this tangent by m, we shall have

$$m = \frac{dx}{dy},$$

so that, determining, by the differentiation of the equation of the curve, the value of $\frac{dx}{dy}$ and making it equal to m, we shall have an equation, which, after having substituted therein for y its value in terms of x and constant quantities, deduced from the equation of the curve, will give the values of x answering to the points where the curve makes such an angle with the ordinate; and if the curve no where makes with the ordinate an angle equal to the given angle, the value or values of x will be imaginary, or the equation will indicate a manifest absurdity. For example, in the hyperbola, having for its equation $y^2 = 2\,(a\,x + x^2)$, we should have

$$\frac{dx}{dy} = \frac{2\,y}{2\,a + 4\,x},$$

which, being equal to m, gives

$$\frac{2\,y}{2\,a + 4\,x} = \frac{y}{a + 2\,x} = m;$$

whence we deduce

$$y = m\,a + 2\,m\,x;$$

but the equation of the curve gives

$$y = \sqrt{2\,(a\,x + x^2)};$$

therefore $\quad m\,a + 2\,m\,x = \sqrt{2\,(a\,x + x^2)},$

or, squaring both members,

$$m^2\,a^2 + 4\,m^2\,a\,x + 4\,m^2\,x^2 = 2\,a\,x + 2\,x^2.$$

If it is asked, now, in what place this hyperbola makes with the

ordinate an angle of 45°; as the tangent of 45° is equal to radius, we shall have $m = 1$, which reduces the equation to

$$a^2 + 4ax + 4x^2 = 2ax + 2x^2$$

or

$$2x^2 + 2ax + a^2 = 0,$$

which being resolved, gives

$$x = -\tfrac{1}{2}a \pm \sqrt{\tfrac{1}{4}a^2 - \tfrac{1}{2}a^2} = -\tfrac{1}{2}a \pm \sqrt{-\tfrac{1}{4}a^2};$$

these values are imaginary, and show that the hyperbola, whose particular equation is

$$y^2 = 2(ax + x^2),$$

no where makes with the ordinate an angle of 45°.

Although the methods employed conformably to the above remark, and in the resolution of the two preceding problems, seem equally applicable to curves of which we have only the differential equation; yet on close examination it will be found, that by this last equation the question cannot be perfectly satisfied by the calculus, as in the case where we have the equation in finite terms. Indeed, when the equation, which resolves the question, still contains x and y, after the substitution of the value of $\frac{dx}{dy}$ the differential equation of the curve will not serve to eliminate y, since, by supposition, it contains dx and dy. And even if it should contain only x, we could not be sure that this value of x would satisfy the question. To be sure of this, it would be necessary to substitute this value of x in the equation of the curve, and deduce thence a real value for y. But we cannot, in any way, deduce from the equation in question, the value of y.

The only means of resolving these questions, in such a case, is, supposing the curve already constructed, to construct also the equation we have obtained, expressing the conditions of the question, which gives a second line, whose intersection or intersections with the first, furnish the solution or solutions required.

33. We may also, by the same principles, determine the rectilineal asymptotes of curved lines.

A curve has a rectilineal asymptote, when, having some branch which is infinitely extended, the tangent at the extremity of this branch meets the axis of the abscissas or that of the ordinates at a finite distance from their origin. Thus (*fig.* 6) if, from the subtangent $PT = \frac{y\,dx}{dy}$ we take the abscissa $AP = x$, we shall have $\frac{y\,dx}{dy} - x$ for the value of AT, or the distance of the origin A from the intersection of the tangent. After having calculated, therefore, the value of $\frac{y\,dx}{dy} - x$, we have only to represent it by a finite quantity s; by means of this equation combined with that of the curve, we may eliminate y or x, and we shall have s in terms of x or y. Then supposing y or x infinite, if there result one or more finite

values for s, they will be distances AC from the origin to the intersections of the axis and the asymptotes. But as a single distance does not fix the positions, we imagine, through the origin A, a straight line AK to be drawn parallel to the ordinates, and observe that the similar triangles TPM, TAK, give

$$TP : PM :: TA : AK;$$

that is to say,

$$\frac{y\,dx}{dy} : y :: \frac{y\,dx}{dy} - x : AK = y - \frac{x\,dy}{dx}.$$

We therefore calculate, in the same way, the value of $y - \frac{x\,dy}{dx}$, and having supposed it equal to a finite quantity t, by means of this equation and that of the curve, we eliminate x or y; and, supposing the remaining x or y infinite, the value or values of t which result, will give the distances AK, which, with the distances AC, determine the position of the asymptotes.

For example, if the equation of the curve were $y^3 = x^2(a+x)$, we should have

$$3y^2\,dy = 2x\,dx\,(a+x) + x^2\,dx = 2ax\,dx + 3x^2\,dx,$$

then

$$\frac{y\,dx}{dy} - x = \frac{3y^3}{2ax+3x^2} - x,$$

or substituting for y^3 its value,

$$\frac{y\,dx}{dy} - x = \frac{3ax^2+3x^3}{2ax+3x^2} - x = \frac{ax^2}{2ax+3x^2} = \frac{ax}{2a+3x} = s;$$

then supposing x infinite, that is to say, neglecting $2a$ by the side of $3x$, we shall have $s = \frac{a}{3}$. We find, in like manner,

$$y - \frac{x\,dy}{dx} = y - \frac{2ax^2+3x^3}{3y^2} = \frac{3y^3 - 2ax^2 - 3x^3}{3y^2},$$

which, if we substitute for y its value, is reduced to

$$\frac{ax^2}{3x\sqrt[3]{(a+x)^2x}} = t;$$

supposing then x to be infinite, we have $t = \frac{1}{3}a$.

If, the value of s being finite, we find that of t infinite, it would prove that the asymptote is parallel to the ordinates. If, on the contrary, s being infinite, t were finite, or zero, or infinitely small, the asymptote would be parallel to the abscissas.

34. In all which precedes, we have supposed that the ordinates were parallel, and moreover that they all issued from the line on which the abscissas are reckoned. But it often happens that we make the ordinates issue from a fixed point. Sometimes we take as abscissas the arcs of a curved line, and as ordinates, straight or curved lines. But, in general, to whatever lines the points of the principal curve be referred, we always have or may have an equation which expresses the ratio of the abscissas to the ordinates. When we wish to make use of it to determine the tangents or other lines, we must take care that the lines we employ to determine these tangents contain

no other differentials than those of the variables which enter into the equation of the curve. We shall now illustrate this by some examples.

35. Let us suppose, first, that AM (*fig.* 9), being a known curve of which we know how to draw the tangents, BS were a curve having for its abscissas the arcs AM of the first, and for its ordinates the lines MS parallel to a given line; the ratio of AM to MS being expressed by any equation, it is required to draw a tangent to the point S of the curve BS.

We imagine, as before, the infinitely small arc Ss, of which the prolongation SQ, or the tangent, meets at Q the tangent MT to the corresponding point M of the curve AM; and, having drawn the line Sk parallel to MT or Mm, the triangle Sks will be similar to the triangle QMS, so that we shall have

$$sk : Sk :: MS : MQ;$$

now if we call the arc AM, x; and the ordinate MS, y; we shall have $Mm = Sk = dx$, and $sk = sm - SM = dy$;

then $$dy : dx :: y : MQ = \frac{y\,dx}{dy};$$

taking then upon the tangent MT the part MQ equal to the value of $\frac{y\,dx}{dy}$ determined by the differentiation of the equation of the curve, we shall have the point Q, through which and the point S, drawing QS, this line will be the tangent required.

Let us suppose, for example, that the curve BS is described so that the ordinate MS is always equal to a determinate part of the arc AM; that is to say, that MS is always to AM in the given ratio of a to b, we should then have $y : x :: a : b$, and the equation of the curve is $by = ax$. Differentiating, $b\,dy = a\,dx$, and consequently $\frac{dx}{dy} = \frac{b}{a}$; then $\frac{y\,dx}{dy}$ or $MQ = \frac{by}{a}$; but by the equation of the curve, $\frac{by}{a} = x$; therefore $MQ = x$. Thus in all the curves whose parallel ordinates have always the same ratio to their corresponding abscissas, whether straight lines or curves, the subtangent QM will always be equal to the corresponding abscissa AM.

36. When the curve AM, upon which we take the abscissas, is a circle; and the ordinate MS is always to the arc AM in a constant ratio, the curve BS is called a *cycloid*. If the ordinate MS is always equal to AM, it is the *common cycloid*, or that traced by a point in the circumference of a circle revolving on a plane. If MS is greater than AM, but still having a constant ratio to it, it is called a *prolate cycloid;* if, on the contrary, MS is less than AM, it is called a *curtate cycloid.*

37. If the equation of the curve to which it is required to draw a tangent, instead of expressing the ratio of AM to MS, expressed that of AM to PS; that is, if the arcs AM were the abscissas x, and the ordinates PMS, or y, were reckoned from a determinate straight line AP; then Su being drawn parallel to AP, the subtangent PI would be determined on the line AP, in the following manner.

As the curve AM is supposed to be known, the subtangent and

tangent to each point of it, are also considered as known; so that, making $PT = s$, and $TM = t$, we shall have, by drawing Mr parallel to AP, and comparing the similar triangles TPM and Mrm,

$$TP : TM :: Mr : Mm; \text{ that is, } s : t :: Mr : dx;$$

therefore $Mr = \frac{s\,dx}{t} = Su$, supposing Su to be parallel to AP. Again; the similar triangles Sus and IPS give

$$us : Su :: PS : PI;$$

and as PS is y, us is dy; therefore

$$dy : \frac{s\,dx}{t} :: y : PI = \frac{s\,y\,dx}{t\,dy}.$$

Then, differentiating the equation of the curve, we find the value of $\frac{dx}{dy}$, which being substituted in $\frac{s\,y\,dx}{t\,dy}$, will give the value of PI, freed from differentials.

38. Sometimes the equation of the curve is not given by the ratio of the abscissas to the ordinates, but by that which each ordinate of the curve is supposed to have to the corresponding ordinates of some other known curve. In that case the tangents are drawn by the following method. Let us suppose, for example, that the curve BMV (*fig.* 10) depends on the two known curves AL and CN, by means of an equation between the corresponding ordinates PL, PM, PN, which we shall call respectively x, y, z. Since the curves AL and CN are supposed to be known, their subtangents PS and PR are also considered as known. Call PS, s and PR, s'; and imagine the ordinate $p\,n\,m\,l$ to be drawn infinitely near to PL, and Lu, Mr, no parallel to CPA. The similar triangles LPS, Lul give

$$PS : PL :: Lu : ul; \text{ that is, } s : x :: Lu : dx;$$

therefore $Lu = \frac{s\,dx}{x} = Mr$. Now the similar triangles TPM, Mrm give $rm : Mr :: PM : PT$; that is,

$$dy : \frac{s\,dx}{x} :: y : PT; \text{ therefore } PT = \frac{s\,y\,dx}{x\,dy};$$

then, if the equation of the curve contained only x and y, we should, by differentiating this equation, have the value of $\frac{dx}{dy}$, which being substituted in $\frac{s\,y\,dx}{x\,dy}$, would give the value of PT freed from differentials. But as this equation contains x, y, and z, its differential will contain dx, dy, and dz; we must therefore find the value of dz expressed in terms of dx and dy. Now the similar triangles Non and NPR give $No : on$ or $Lu :: NP : PR$; that is, observing that while PM increases, PN diminishes, so that its difference No or dz is negative,

$$-dz : \frac{s\,dx}{x} :: z : s'; \text{ therefore } dz = \frac{-s\,z\,dx}{s'\,x};$$

so that, putting in the differential equation of the curve, the quantity $\frac{-s\,z\,dx}{s'\,x}$, instead of dz, we shall easily find the value of $\frac{dx}{dy}$ to be

substituted in the formula $\frac{s y d x}{x d y}$ of the subtangent. As an example, let us suppose that AL and CN being any two known curves, the ordinate PM is always a mean proportional between PL and PN, we have $x : y :: y : z$; or $x z = y^2$ for the equation of the curve BM. Differentiating, it becomes

$$x\,dz + z\,dx = 2y\,dy;$$

substituting for dz its general value $-\frac{s z d x}{s' x}$, we have

$$-\frac{s z d x}{s'} + z\,dx = 2y\,dy,$$

whence we deduce $\frac{dx}{dy} = \frac{2 s' y}{z(s' - s)}$; then $PT = \frac{s y d x}{x d y}$ becomes $\frac{2 s s' y^2}{x z (s' - s)}$, or, substituting for y^2 its value $x z$, and reducing,

$$PT = \frac{2 s s'}{s' - s}.$$

It is easy to vary these examples by taking any equation we please in terms of x, y, and z. We may, if we please, suppose AL and CN to be straight lines (*fig.* 11). In this case, taking always PM a mean proportional between PL and PN, the curve BM is a conic section, viz. a parabola, when the point C is infinitely distant, or the straight line CN is parallel to AC; an ellipse, when the two angles HAC and HCA are acute; and, particularly, a circle, when they are each 45°; and an hyperbola, when one of the two angles is obtuse.

39. When the ordinates issue from a fixed point, we take as abscissas the arcs of a known curve, which most commonly is a circle; that is, in this last case, the equation expresses the ratio of the ordinate CM (*fig.* 12) to the angle ACM, which that line makes with another, such as AC, given in position; or it announces the ratio of the ordinate CM to the arc OS described with a determinate radius.

To draw the tangents, when we have the equation between the ordinate CM and the angle ACM, or the arc OS, we imagine that, for each point M, there is raised upon CM a perpendicular CT which meets the tangent TM in T, then taking the infinitely small arc $M m$, and drawing the ordinate $m C$, we conceive that with the radius CM, there has been described the arc Mr, which may be considered as a straight line perpendicular to Cm at r. As the angle $M m r$ differs infinitely little from the angle TMC, the triangles $M r m$ and TCM are similar, and give $r m\ (dy) : M r :: C M\ (y) : CT$; or

$$dy : M r :: y : CT = \frac{M r \times y}{dy};$$

calling the arc OS, x, and its radius, a, the similar sectors CSs and CMr give $CS : CM :: S s : M r$; that is,

$$a : y :: dx : M r = \frac{y\,dx}{a}.$$

Substituting this value of $M r$ in that of CT, we have CT or the

subtangent $= \frac{y^2\,dx}{a\,dy}$. Now as the ratio between x and y is supposed to be known, it will be easy, by differentiating the equation which expresses this ratio, to obtain the value of $\frac{dx}{dy}$, which, being substituted in that of CT, will give a new expression for CT, freed from differentials.

If we suppose, for example, that the ordinate CM (*fig.* 13) is always to the corresponding arc OS, in the ratio of m to n, that is, if $y : x :: m : n$, we have $n\,y = m\,x$; then $n\,dy = m\,dx$, and consequently $\frac{dx}{dy} = \frac{n}{m}$; therefore

$$CT = \frac{y^2\,dx}{a\,dy} = \frac{y^2}{a} \times \frac{n}{m} = \frac{y}{a} \times \frac{ny}{m};$$

now, by the equation, we have $\frac{ny}{m} = x$; therefore $CT = \frac{yx}{a}$. If, then, from the point C, as a centre, with the radius CM, we describe the arc MQ, we shall have $CT = MQ$. For the similar sectors COS and CQM give $CS : OS :: CM : MQ$, that is,

$$a : x :: y : MQ = \frac{yx}{a}; \text{ therefore } CT = MQ.$$

The curve of which we have just been speaking is the *spiral of Archimedes.*

40. Let us now suppose that, OS (*fig.* 14) being a known curve, or one whose tangents may be drawn, the curve BM is constructed with this condition, that CS, x, and CM, y, have with each other a determinate ratio expressed by a given equation. If we conceive the infinitely small arc Mm, the ordinates CM, Cm, and the arcs Mr, Sq, described from the centre C and with the radii CM and CS, it is evident that the differentiation of the equation will give only the ratio of dy to dx or rm to sq, since y and x are the only variables which enter into this equation. But, in order to determine the subtangent CT, we must have the ratio of rm to rM. Now rM may be thus determined by means of the conditions of the question.

Since the curve OS is known, the subtangent CQ for each point S is given; now the similar triangles QCS, Sqs give

$$CS : CQ :: qs : qS;$$

calling, therefore, CQ, s, we shall have $x : s :: dx : qS = \frac{s\,dx}{x}$;

but the similar sectors CSq, CMr, give

$$CS : CM :: Sq : Mr; \text{ or } x : y :: \frac{s\,dx}{x} : Mr = \frac{s\,y\,dx}{x^2};$$

it is now therefore easy to find the subtangent CT, by comparing the similar triangles Mrm and TCM, which give

$$rm : rM :: CM : CT, \text{ or } dy : \frac{s\,y\,dx}{x^2} :: y : CT = \frac{s\,y^2\,dx}{x^2\,dy}.$$

To apply this, let us suppose the curve BM to be constructed by taking SM always of the same magnitude, or equal to a given line

a, whatever the line OS may be. We shall have then, $x + a = y$. Therefore $dx = dy$, and consequently $\frac{dx}{dy} = 1$; the subtangent CT becomes then $CT = \frac{sy^2}{x^2}$, which is constructed in the following manner.

Through the point M and parallel to the tangent SQ, we draw a line MN; then having joined the two points S and N, we draw from the same point M a line MT parallel to SN, and MT, thus determined, will be the tangent required. For the similar triangle CSQ, CMN, give $CS : CQ :: CM : CN$, or $x : s :: y : CN = \frac{sy}{x}$. In like manner, the similar triangles CSN and CMT, give

$$CS : CN :: CM : CT; \text{ or } x : \frac{sy}{x} :: y : CT = \frac{sy^2}{x^2}$$

We perceive, by these examples, how to proceed in the application of the same methods to other cases. It is to be observed in conclusion, that when OS is a straight line, the curve BM, which is formed by taking SM always of the same magnitude, is what is called the *Conchoid of Nicomedes.*

Application to the limits of curved lines, and in general to the limits of quantities, and to questions on Maxima *and* Minima.

2

41. We have seen (32), that $\frac{dx}{dy}$ expresses the tangent of the angle which the curve, or its tangent makes, at each point, with the ordinate; and that $\frac{dy}{dx}$ represents that of the angle which the curve or its tangent makes with the axis of the abscissas.

To know, therefore, at what point the tangent of a curve becomes parallel to the ordinates, we must find the values of x and y corresponding to $\frac{dx}{dy} = 0$, or simply to $dx = 0$; and to find where the tangent of the curve is parallel to the abscissas, we must suppose $\frac{dy}{dx} = 0$, or merely $dy = 0$, and the values of x and y which result, are those of the point of contact.

It evidently follows from this that, in order to find whether a curve, of which we know the equation, has, at any point, its tangent parallel to the ordinates or to the abscissas, it is necessary to differentiate the equation, and having deduced the value of $\frac{dx}{dy}$, if we make the numerator of this ratio equal to zero, we shall

have an equation which, with the equation of the curve, will give the value of x and that of y, which determine at what points the tangent is parallel to the ordinates; so that if there are more than one of these points we shall have several values for x and y.

On the contrary, if we make the denominator equal to zero, this equation, conjointly with that of the curve, will determine the values of x and y, which answer to the points where the tangent of the curve becomes parallel to the abscissas. It must however be observed, that although $d\,x$ is always zero, when the tangent is parallel to the ordinates, as well as $d\,y$, when the tangent is parallel to the abscissas, still it must not be concluded, when the value or values of x resulting from the supposition that $d\,x = 0$, or that $d\,y = 0$, are found, that the tangent is parallel to the ordinates or abscissas, except when we have not at the same time $d\,x = 0$, and $d\,y = 0$.

To illustrate these rules by a familiar example, let us take the curve which has for its equation

$$y^2 + x^2 = 3\,a\,x - 2\,a^2 + 2\,b\,y - b^2,$$

which, on the supposition that x and y are perpendicular to each other, belongs to the circle, the origin of the coördinates being at the point A.

The lines AP (*fig.* 15) are x, and the lines PM, PM' are the two values of y which the resolution of the equation gives for each value of x.

If we differentiate this equation, we have

$$2\,y\,d\,y + 2\,x\,d\,x = 3\,a\,d\,x + 2\,b\,d\,y,$$

whence we deduce $$\frac{d\,x}{d\,y} = \frac{2\,y - 2\,b}{3\,a - 2\,x}.$$

Let us first make the numerator equal to zero; to find the points at which the tangent becomes parallel to the ordinates, we shall have $2\,y - 2\,b = 0$, or $y = b$. Substituting this value in the equation of the curve, it becomes,

$$b^2 + x^2 = 3\,a\,x - 2\,a^2 + 2\,b^2 - b^2, \text{ or } x^2 - 3\,a\,x = -2\,a^2,$$

which, being resolved, gives $x = \frac{3}{2}\,a \pm \sqrt{\frac{1}{4}\,a^2}$; that is, $x = 2\,a$, and $x = a$; which shows that the curve or its tangent becomes parallel to the ordinates at the two points R and R', which have for the ordinate the line b, and of which one, R, has for its abscissa the line $AQ = a$, and the other, R', has for its abscissa the line $AQ' = 2\,a$.

Let us now make the denominator of $\frac{dx}{dy}$ equal to zero, to find at what point the curve or its tangent becomes parallel to the abscissas. We shall have $3a-2x=0$, or $x=\frac{3}{2}a$. Substituting this value in the equation of the curve, we have

$$y^2+\frac{9}{4}a^2=\frac{9}{2}a^2-2a^2+2by-b^2,$$

or

$$y^2-2by+b^2=\frac{1}{4}a^2;$$

and, extracting the square root, $y-b=\pm\frac{1}{2}a$; therefore

$$y=b\pm\frac{1}{2}a;\text{ that is, } y=b+\frac{1}{2}a, \text{ and } y=b-\frac{1}{2}a,$$

which shows that the tangent becomes parallel to the abscissas, at the two points T and T', which have as their common abscissa the line $AS=\frac{3}{2}a$, and of which, T' has for its ordinate

$$ST'=b+\frac{1}{2}a, \text{ and } T, \text{ the line } ST=b-\frac{1}{2}a.$$

The points Q and Q' are called the limits of the abscissas, because, between Q and Q' for each abscissa AP, are corresponding real values PM and PM' for y; while between Q and A, and beyond Q' with reference to A, there is no point of the curve, so that if x be supposed smaller than AQ or a, or greater than AQ' or $2a$, no real value can be found for y. Indeed, if in the equation we substitute for x a quantity $a-q$ smaller than a, or a quantity $2a+q$ greater than $2a$, on resolving the equation, the two values of y will be found to be imaginary.

In like manner, if through the point A we conceive AL' to be drawn parallel to the ordinates, that is, to the axis of the ordinates; and if through the points T and T', the lines TL, $T'L'$ be drawn parallel to the abscissas; the lines

$$AL=ST=b-\frac{1}{2}a, \text{ and } AL'=ST'=b+\frac{1}{2}a,$$

are the limits of the ordinates; for it is evident that there can be no ordinate greater than AL', nor smaller than AL, the tangent being supposed parallel to the abscissas. In fact, if in the equation of the curve, a quantity, $b-\frac{1}{2}a-q$ smaller than $b-\frac{1}{2}a$, be substituted for y, it will be found, on resolving the equation, that the values of x are imaginary. The same thing will happen, if instead of y, the quantity $b+\frac{1}{2}a+q$, greater than $b+\frac{1}{2}a$ be substituted.

42. The ordinate ST' is the greatest of all those which terminate in the concave part $RT'R'$ of the circumference. The ordinate ST is the least of those which terminate in the convex part; and the ordinates QR, $Q'R'$ are, at once, the least for the concave part, and the greatest for the convex.

43. Thus the same method serves at the same time, 1st, to assign the limits of the abscissas and ordinates; 2d, to determine in what cases the tangent becomes parallel to the abscissas or to the ordinates; 3d, to find the greatest and the least abscissas or ordinates.

44. Now in whatever manner a quantity is expressed algebraically, the algebraical expression which represents it, may always be considered as the expression for the ordinate of a curved line. For example, if $\frac{x^2 \times (a-x)}{a^2}$ is the expression of a quantity which we call y, in which case we have $y = \frac{x^2 \times (a-x)}{a^2}$, we may consider this equation as that of a curved line whose abscissa is x, and ordinate, y. If then the quantity $\frac{x^2 \times (a-x)}{a^2}$ may, in a certain case, become greater or smaller than in any other case (which is called being susceptible of a *maximum* or *minimum*), it is evident that we must pursue exactly the same method as above, that is, differentiate the equation, and having deduced from it the value of $\frac{dx}{dy}$, make the numerator or denominator of that value equal to zero.

45. It is to this that the method which is called that of *maxima* and *minima* is reduced. This method is one of the most useful in analysis, and has for its object to find, among several quantities which increase or decrease according to a certain law, that which is greatest or least; or, in general, that, among all similar quantities, which possesses certain properties in the highest degree. We shall now give some examples from Geometry and Algebra. Mechanics will hereafter furnish some at once more curious and more useful.

46. Let it be proposed to divide a given number † a into two parts, such that their product shall be a *maximum.* Call x one of the parts; the other will be $a - x$, and the product will be represented by $a x - x^2$. Let this product $= y$; we have $y = a x - x^2$; by differentiating, therefore, $d y = a d x - 2 x d x$, and consequently $\frac{d x}{d y} = \frac{1}{a - 2 x}$; if we make the numerator equal to zero,

† See the Introduction.

we have $1 = 0$, which is absurd; consequently if there be a maximum, it will be shown by making the denominator equal to zero; let therefore $a - 2x = 0$, whence $x = \frac{1}{2} a$, which shows, that among all the different ways in which a number may be divided into two parts, the product of the two parts will be greatest, when each is one half of the number.†

47. When, as in this example, we have the algebraical expression of a quantity, we may dispense with making it equal to a new variable y; we have merely to differentiate, and make the numerator or denominator, if the differential is a fraction, equal to zero. Thus, in the same example, we merely differentiate $a x - x^2$, and, making the differential $a\,d\,x - 2\,x\,d\,x$ equal to zero, we have $a\,d\,x - 2\,x\,d\,x = 0$, whence we deduce, as before $x = \frac{1}{2} a$.

48. To take a more general question, let it be proposed to divide a known number a into two parts, such that the product of a determinate power of one of the parts by the same or another power of the other part, shall be the greatest possible. Let x represent the first part, and m the power to which it is to be raised; the second part will be $a - x$, and if n designate its power, the product in question will be $x^m (a - x)^n$. If we differentiate this product and make the differential equal to zero, we shall have

$$m\,x^{m-1} d\,x\,(a - x)^n - n\,x^m d\,x\,(a - x)^{n-1} = 0.$$

Dividing the whole by $x^{m-1} d\,x\,(a - x)^{n-1}$, we have

$$m\,(a - x, - n\,x = 0, \text{ or } m\,a - m\,x - n\,x = 0,$$

which gives $x = \frac{m\,a}{m + n}$. If we suppose, for example, that it is required to divide a into two such parts that the square of one multiplied by the cube of the other part, shall be the greatest product possible; then $m = 2$, $n = 3$. We have therefore

$$x = \frac{2\,a}{2 + 3} = \frac{2}{5}\,a;$$

that is, that one of the parts must be $\frac{2}{5}$ of the number or quantity proposed, and the other be consequently three fifths.

What has been said above in relation to *figure* 15, shows that a quantity may become greatest among all quantities of the same

† It is here to be observed that those considerations, employed in the solution of this question in the introduction, do not here appear, but are contained in the calculus, being principles which are at the foundation of this mode of considering questions.

kind, in two different ways; either when increasing first, like PM, it diminishes afterwards, or when, like $P'M'$, it continues to increase, until it stops suddenly on becoming $Q'R'$; but in this last case, it is at once the greatest of all the ordinates which terminate in the convex part, and the least of those which terminate in the concave part. In like manner a quantity may become least of all those of the same kind, in two different ways; either, like PM, by diminishing at first to increase afterwards, or, like $P'M''$, by diminishing until it suddenly stops, and then it is at once a *minimum* and a *maximum*; it is a *minimum* with reference to the branch $MT'M''$, and a *maximum* with reference to the branch MTM'.

49. So that, to ascertain whether a quantity is a *maximum* or a *minimum*, or both, we must, supposing a to mark the value of x, corresponding to the *maximum* or *minimum*, substitute successively instead of x, in the quantity proposed, $a+q$, a, and $a-q$. If the two extreme results are real and less than the middle one, the quantity is a *maximum*; if, on the contrary, the intermediate result is the least, the quantity is a *minimum*; finally, if of the two extreme results, one is imaginary and the other real, the quantity is at once a *maximum* and a *minimum*.

50. When, in the determination of a *maximum* or *minimum*, the value found for the variable renders that of the *maximum* or *minimum* negative, we must conclude that the *maximum* or *minimum* which it indicates, does not belong to the question under consideration, but that it answers to a question in which some of the conditions are of a contrary character. If, for example, it were required to divide the line AB (*fig.* 16) at the point C, in such a manner that the square of the distance AC from the point A, being divided by the distance BC from the point B, gives the least possible quotient; then, calling the given line AB, a, and the distance AC, x; the remainder CB is $a-x$, and consequently the quotient is $\frac{x^2}{a-x}$; differentiating this quantity or $x^2(a-x)^{-1}$, we have $2x\,dx\,(a-x)^{-1}+x^2\,dx\,(a-x)^{-2}=0$,

or
$$\frac{2x\,dx}{a-x}+\frac{x^2\,dx}{(a-x)^2}=0,$$

or $2ax\,dx-x^2\,dx=0$, or $(2a-x)x=0$;

which gives either $x=0$, or $2a-x=0$; the first value indicates

a *minimum*, which is evident without calculation. As to the second, which gives $x = 2a$, if we substitute it in $\frac{x^2}{a-x}$, we find $\frac{4a^2}{-a}$ or $-4a$. The *minimum*, therefore, does not belong to the present question. But if we examine the value of $x = 2a$ just obtained, we see that the point C cannot be between A and B, but that the question will have a solution, if it be required to find it in AB produced beyond B with regard to A. In that case, if we call AC', x, the distance BC' will not be $a - x$, but $x - a$, and the quantity under consideration will be $\frac{x^2}{x-a}$, which, being differentiated and made equal to zero, gives

$$\frac{2x\,dx}{x-a} - \frac{x^2\,dx}{(x-a)^2} = 0,$$

or, after making the reductions,

$$x^2\,dx - 2ax\,dx = 0,$$

which gives $x = 2a$, as before; but this quantity, substituted in $\frac{x^2}{x-a}$, changes it into $4a$. There is therefore a *minimum* for this case.

If the denominator $x - a$ of the differential be made equal to zero, we have $x = a$, which indicates a *maximum*; and indeed, when $x = a$, the quantity becomes infinite. But it has nevertheless the true character of a *maximum*, for whether we suppose x greater or less than a, it gives a less quantity than to suppose $x = a$.

51. When the expression of a quantity of which we wish to know the *maximum* or *minimum*, contains any constant multiplier or divisor, we may suppress this multiplier or divisor before differentiating; for if we suppose that $\frac{ay}{b}$ represents a quantity which is to be a *maximum*, or a *minimum*, a and b being constant, $\frac{a\,dy}{b}$ must be $= 0$; but since a and b are not zero, dy must necessarily be $= 0$; the conclusion is therefore the same as if the *maximum* of y only were required, that is, the same as if the constant divisors and factors were suppressed. This remark serves, in many cases, to simplify the calculation.

52. Let it now be proposed to find, among all the lines which may be drawn through the same point D given in the angle

ABC (*fig.* 17), that which forms with the sides of the angle the least possible triangle.

Through the point D let DG be drawn parallel to the side AB, and supposing EF any straight line drawn through the point D, let fall upon BC the perpendicular DK, and from the point E, where EF meets AB, let fall upon BC the perpendicular EL. The line BG is considered as known, as well as the perpendicular DK; let therefore $BG = a$, $DK = b$, and let BF, the base of the triangle BEF, be $= x$. We see that, from a certain point, the more BF increases, the greater will be the triangle. If, on the contrary, BF diminishes, the triangle will diminish also, but only to a certain point; for if BF should become nearly equal to BG, the straight line EDF would be nearly parallel to AB, since it would nearly coincide with GD, in which case the triangle would be very great. There is then a certain value of BF, which gives the smallest triangle possible. In order to find this value, we must discover the general expression of the triangle BEF. Now the similar triangles BEF, GDF give

$$GF : BF :: DF : EF;$$

and the similar triangles DKF and ELF, give

$$DF : EF :: DK : EL;$$

therefore $GF : BF :: DK : EL$; that is,

$$x - a : x :: b : EL = \frac{bx}{x-a};$$

therefore the surface of the triangle BEF, which is $\frac{EL \times BF}{2}$, will be $\frac{bx}{x-a} \times \frac{x}{2}$, or $\frac{\frac{1}{2}bx^2}{x-a}$. We must therefore differentiate this quantity, and make the numerator or denominator equal to zero, or, since we may suppress the constant factor $\frac{1}{2}b$ (51), we need only differentiate the quantity $\frac{x^2}{x-a}$; and, not to repeat an operation which has been already performed (50), we shall find $x = 2a$; if, therefore, we take $BF = 2a = 2\,BG$, the line FDE, drawn through the point D, will give the triangle FBE for the *minimum* required.

53. Let it be proposed to find, among all the parallelopipeds of the same surface and the same altitude, that which has the greatest capacity.

Call h the altitude and c^2 the surface of the parallelopiped, x and y the two sides of the rectangle which serves as base. The whole surface is composed of six rectangles, of which two have each x for their base and h for their altitude, two others have h for their altitude and y for their base, and the two last have x for base and y for altitude; so that the whole surface is expressed by

$$2hx + 2hy + 2xy;$$

that is, we have

$$2hx + 2hy + 2xy = c^2.$$

The capacity or solidity is hxy. Since then it must be the greatest of all those of the same surface, we have $hx\,dy + hy\,dx = 0$, or, what comes to the same thing,

$$x\,dy + y\,dx = 0.$$

But the equation $2hx + 2hy + 2xy = c^2$, which shows that the surface of all these parallelopipeds is constant or the same, gives $\quad 2h\,dx + 2h\,dy + 2x\,dy + 2y\,dx = 0$

Substituting, therefore, in this equation, the value of $d\,x$ found in the other, we have, after making the reductions, $y = x$; the base therefore must be a square. To find the value of the side, we substitute for y its value x in the equation

$$2hx + 2hy + 2xy = c^2,$$

which thereby becomes $4hx + 2x^2 = c^2$; which, being resolved, gives $x = -h \pm \sqrt{h^2 + \frac{1}{2}c^2}$, whose root $x = -h - \sqrt{h^2 + \frac{1}{2}c^2}$, being negative, does not belong to the present question; thus the true value of x is

$$x = -h + \sqrt{h^2 + \tfrac{1}{2}c^2}.$$

54. If it is now asked what must be the altitude h, in order that the parallelopiped may have the greatest solidity among all those of the same surface; we observe that since, the altitude being h, the base must be a square, this solidity will be expressed by hx^2; considering then h and x as variable, the differential of hx^2 must be equal to 0; we have therefore

$$2hx\,dx + x^2\,dh = 0, \text{ or } 2h\,dx + x\,dh = 0.$$

But the equation $4hx + 2x^2 = c^2$, which indicates that the surface is constant, gives for its differential,

$$4h\,dx + 4x\,dh + 4x\,dx = 0;$$

substituting, therefore, in this equation, for dh, its value found from the equation $2h\,dx + x\,dh = 0$, and making the reductions, we have $h = x$; the parallelopiped sought must therefore be a

cube, since its altitude h is equal to the side x of the square which serves as base. To find now the value of the side of this cube, we must substitute for h its value x, in the equation

$$4hx + 2x^2 = c^2,$$

which thus becomes

$$4x^2 + 2x^2 = c^2, \text{ or } 6x^2 = c^2,$$

which gives $x = \sqrt{\frac{cc}{6}}$.

Therefore, of all the parallelopipeds of the same surface, that which has the greatest solidity is the cube which has for its side a line equal to the square root of the sixth part of that surface.

55. Let it now be required to find, among all the triangles of the same perimeter and same base, that which has the greatest surface.

Let a be the base AB, and c the perimeter of the triangle ABC (*fig.* 18). Let fall the perpendicular CP, and call AP, x; CP, y; we shall have

$$PB = a - x;\ AC = \sqrt{x^2 + y^2},$$

and

$$CB = \sqrt{y^2 + (a-x)^2}.$$

Then the perimeter will be

$$\sqrt{x^2+y^2} + \sqrt{y^2+(a-x)^2} + a = c,$$

and the surface $= \frac{ay}{2}$.

Now, by the conditions of the question, we must have

$$\frac{a\,dy}{2} = 0, \text{ or } dy = 0.$$

But, if we differentiate the expression for the perimeter, we have

$$\frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} + \frac{-dx\,(a-x) + y\,dy}{\sqrt{y^2+(a-x)^2}} = 0.$$

which, dy being $= 0$, is reduced to

$$\frac{x\,dx}{\sqrt{x^2+y^2}} - \frac{(a-x)\,dx}{\sqrt{y^2+(a-x)^2}} = 0;$$

or, dividing by dx, and freeing from fractions,

$$x\sqrt{y^2+(a-x)^2} - (a-x)\sqrt{y^2+x^2} = 0.$$

Squaring the two members, performing the operations indicated, suppressing the terms which cancel each other, and reducing, we arrive at this equation,

$$x^2 = (a-x)^2 = a^2 - 2ax + x^2;$$

whence $x = \frac{1}{2}a$, which shows that the triangle must be isosceles.

To construct it, we raise a perpendicular from the middle of AB, and having described from the point B as a centre, with a radius equal to half the excess of the perimeter c over the base a, an arc cutting that perpendicular in C, and drawn CB and CA, we have the triangle which has the greatest surface among all those of the same perimeter, and the same base.

56. If it is now required to find, generally, among all the triangles of the same perimeter, that which has the greatest surface, it must be observed, that whatever be the base, we see by the preceding solution that x must always be equal to half of it, that is, that whatever be the values of a, x is always $= \frac{1}{2} a$. This being the case, the equation which expresses the perimeter will be reduced to $\sqrt{\frac{1}{4} a^2 + y^2} + \sqrt{\frac{1}{4} a^2 + y^2} + a = c$,

or $2\sqrt{\frac{1}{4} a^2 + y^2} = c - a$;

squaring and finding the value of y, we have

$$y = \sqrt{\frac{c^2 - 2ac}{4}}.$$

The surface $\frac{ay}{2}$ of the triangle will therefore be

$$\frac{a}{2}\sqrt{\frac{c^2 - 2ac}{4}}.$$

Since then it must be the *maximum* among all those of the same perimeter, whatever may be the base a, we must make equal to zero the differential of

$$\frac{a}{2}\sqrt{\frac{c^2 - 2ac}{4}}, \text{ or of } a\sqrt{c^2 - 2ac},$$

a being considered as variable. We shall have then

$$d \,.\, a\sqrt{c^2 - 2ac} = d \,.\, a\,(c^2 - 2ac)^{\frac{1}{2}}$$

$$= (c^2 - 2ac)^{\frac{1}{2}}\, da - c\,a\,da\,(c^2 - 2ac)^{-\frac{1}{2}}$$

$$= da\sqrt{c^2 - 2ac} - \frac{c\,a\,da}{\sqrt{c^2 - 2ac}} = 0.$$

Making the denominator to disappear,

$$da\,(c^2 - 2ac) - c\,a\,da = 0\,; \text{ whence } a = \frac{c}{3};$$

therefore the base a must be a third of the perimeter; and as we have already found the triangle to be isosceles, it follows that it must be equilateral. Therefore of all the triangles of the same perimeter, the equilateral is that which has the greatest surface.

57. In these two solutions, we have not made the denominator equal to zero, because, in the first, that would have given an imaginary value for x; and in the second, we should have found $a=\frac{1}{2}c$, which would have no better satisfied the question, since, if the base were half of the perimeter, the two other sides would be confounded with the base, and the triangle would be zero. In future, whenever the supposition of the numerator or denominator made equal to zero, would lead to no admissible solution, we shall, to avoid useless investigations, pass over it without notice.

58. In the last question but one, we were able to determine, among all the parallelopipeds of the same surface, that which has the greatest capacity, only by first considering parallelopipeds of the same altitude. In like manner, in the last question, we have found among all triangles of the same perimeter, that, which had the greatest surface, by first resolving the question for triangles of the same base.

It is usually more simple to proceed in this way; that is, to resolve the question by making the least possible number of quantities vary at once, and afterwards successively treating as variable each one of the quantities which have been considered constant. If, for example, it were required to divide a given number into three parts, in such a manner that the product of these three parts should be the greatest possible; calling x and y two of these parts, and a the given number, the third part will be $a-x-y$, and the product of the three will be $xy(a-x-y)$, of which the differential must be made equal to zero. But, instead of considering x and y as both variable at the same time, we shall differentiate, considering x only as variable; we have then,

$$ay\,dx-2xy\,dx-y^2\,dx=0,$$

whence we deduce $\quad x=\frac{1}{2}(a-y).$

The product $xy(a-x-y)$ is thus changed into $\frac{1}{4}y(a-y)^2$. We now differentiate, considering y as variable, and have

$$\tfrac{1}{4}dy(a-y)^2-\tfrac{2}{4}y\,dy(a-y),$$

which we also make equal to zero, and have

$$dy(a-y)^2-2y\,dy(a-y)=0,$$

whence we deduce $\quad y=\frac{1}{3}a;$

wherefore x, and $a-x-y$ are each equal to $\frac{1}{3}a$.

59. We may also, if we please, make all the variable quantities vary together, then, collecting all the terms which are multiplied by the differential of the same variable, make their sum equal to zero, and do the same thing with regard to the differential of each variable. Thus, in the last example, we should have

$$ax\,dy+ay\,dx-2xy\,dx-x^2\,dy-2xy\,dy-y^2\,dx=0,$$

whence we obtain, by making equal to zero the sum of the terms affected by dx, and that of the terms affected by dy,

$$ay\,dx-2xy\,dx-y^2\,dx=0,$$

and $$ax\,dy-2xy\,dy-x^2\,dy=0;$$

or, dividing the first by $y\,dx$, and the second by $x\,dy$,

$$a - 2x - y = 0$$
$$a - 2y - x = 0,$$

equations from which we immediately conclude that $x = \frac{1}{3}a$, and that $y = \frac{1}{3}a$, as before.

It is easy to perceive the reason of this process, if we observe that the only condition to be answered is that the whole differential be equal to zero. Now this condition can be answered generally in one of two ways only, either by supposing each of the two differentials dx and dy equal to zero, which indeed satisfies the equation, but makes nothing known, or by supposing that the sum of the terms multiplying dx, and of the terms multiplying dy, are each zero, which is precisely what we have done.

60. When the conditions of the question are expressed by several equations, we must, before applying this rule to the differential equation which is to determine the *maximum* or *minimum*, deduce from the other differentiated equations the values of the differentials of as many variables as there are equations besides that, and substitute them in that equation; the rule is then to be applied as if there were that equation alone. Thus, in the example already given of the greatest parallelopiped, we had this equation,

$$2hx + 2hy + 2xy = c^2,$$

and the condition that hxy was to be a *maximum*. If then h, x, and y are to be considered as varying at once, the first equation, being differentiated, gives

$$2h\,dx + 2x\,dh + 2h\,dy + 2y\,dh + 2x\,dy + 2y\,dx = 0,$$

and the condition of the *maximum* gives

$$hx\,dy + hy\,dx + xy\,dh = 0.$$

From the first we find

$$dh = \frac{-y\,dx - x\,dy - h\,dy - h\,dx}{x+y}$$

substituting this value in the second equation, we have, after the usual reductions,

$$hx^2\,dy + hy^2\,dx - xy^2\,dx - x^2y\,dy = 0.$$

We may now put equal to zero the sum of the terms multiplying dx, and that of the terms multiplying dy. We have

$$hy^2 - xy^2 = 0 \text{ or } h = x, \text{ and } hx^2 - x^2y = 0 \text{ or } h = y;$$

and since $h = x$, we have also $y = x$; the three dimensions x, y and h are therefore equal, which agrees with the former solution; and putting these values in the equation

$$2hx + 2hy + 2yx = c^2,$$

we have $6h^2 = c^2$, whence

$$h = \sqrt{\frac{cc}{6}},$$

as in the former solution.

61. We may not only make the quantities vary successively, or all at once, but we may take as constant any function whatever of

these quantities, provided that the number of these new arbitrary conditions, united to that of the conditions of the question, be not greater than the number of the variables x, y, z, which enter into the question. This remark may be of the greatest use in many questions, especially when there are radical quantities. For example, let it be required to find among all the quadrilaterals of the same perimeter, that which has the greatest surface. If from the angles C and D (*fig.* 19) we let fall the perpendiculars CF and DE upon the side AB, and from the point D draw DK parallel to AB; then, calling AE, s; DE, t; AF, u; CF, x; and BF, y; we have by the property of right-angled triangles,

$$DA=\sqrt{s^2+t^2},\ DC=\sqrt{(s+u)^2+(x-t)^2},\ CB=\sqrt{x^2+y^2};$$

then if the perimeter be equal to a, we have

$$\sqrt{s^2+t^2}+\sqrt{(s+u)^2+(x-t)^2}+\sqrt{x^2+y^2}+u+y=a.$$

Again, the surface $ABCD$ is equivalent to the trapezoid

$DEFC$ — the triangle DAE + the triangle CFB;

that is,

$$ABCD=\left(\frac{t+x}{2}\right)(s+u)-\frac{st}{2}+\frac{xy}{2}.$$

This being laid down, it would be necessary to differentiate the two preceding equations. But the radical quantities would render the subsequent operations very complicated. To avoid these difficulties, we suppose, at first, that the three radical quantities are constant, which gives

$$d(\sqrt{s^2+t^2})=d(s^2+t^2)^{\frac{1}{2}}=\tfrac{1}{2}d(s^2+t^2)(s^2+t^2)^{-\frac{1}{2}}=\frac{s\,ds+t\,dt}{\sqrt{s^2+t^2}},$$

which being $=0$, because the quantity was considered as constant, we have

1st, $$s\,ds+t\,dt=0.$$

We find also for the differential of the second radical

2d, $$(s+u)(ds+du)+(x-t)(dx-dt)=0;$$

and for the third

3d, $$x\,dx+y\,dy=0.$$

The equation of the perimeter being differentiated on the same supposition, gives (since the differential of each of the radical quantities is zero),

4th, $$du+dy=0.$$

The expression of the surface being differentiated and put equal to zero on account of the condition of its being a *maximum*, gives, 5th,

$$(s+u)(dt+dx)+(t+x)(ds+du)-t\,ds-s\,dt+x\,dy+y\,dx=0,$$

or

$$u\,dt+s\,dx+u\,dx+t\,du+x\,ds+x\,du+x\,dy+y\,dx=0.$$

The first gives $ds=\frac{-t\,dt}{s}$; the third $dx=-\frac{y\,dy}{x}$, and the fourth $du=-dy$. Substituting these values in the second and fifth, we have, after all reductions,

$$-(t\,dt+s\,dy)(u+s)x-(y\,dy+x\,dt)(x-t)s=0,$$

and

$$s u x\, d t - s u y\, d y - s^2 y\, d y - t s x\, d y - x^2 t\, d t - s y^2\, d y = 0.$$

If we deduce from this last the value of $d\, t$, we shall see that all the terms of its numerator are affected by s, and that, by substituting it in the preceding, all the terms will also be affected by s, wherefore $s = 0$, which shows that the angle DAB must be a right angle. This being determined, the equation of the perimeter is reduced to

$$t + \sqrt{u^2 + (x - t)^2} + \sqrt{x^2 + y^2} + u + y = a;$$

and the expression of the surface becomes

$$(t + x)\frac{u}{2} + \frac{x y}{2}.$$

Let us now differentiate, supposing only the two radicals constant, we have

$$u\, d u + (x - t)\,(d x - d t) = 0;$$
$$x\, d x + y\, d y = 0;$$
$$d t + d u + d y = 0;$$
$$u\,(d t + d x) + (t + x)\, d u + x\, d y + y\, d x = 0.$$

The second equation gives

$$d y = -\frac{x\, d x}{y};$$

the third,

$$d t = -d u - d y = \text{(substituting for } d\, y\text{)}\ \frac{x\, d x - y\, d u}{y};$$

these values, substituted in the first and fourth equations, give

$$y u\, d u + (x - t)\,(y\, d x - x\, d x + y\, d u) = 0,$$

and

$$u\,(x\, d x - y\, d u + y\, d x) + (t + x)\, y\, d u - x^2\, d x + y^2\, d x = 0.$$

Now, if, from one of these we find the value of x, and substitute it in the other, we shall have an equation of which all the terms will be affected by y, and which consequently gives $y = 0$, and shows that the angle CBA must also be a right angle. This being the case, the equation of the perimeter becomes

$$t + \sqrt{u^2 + (x - t)^2} + x + u = a,$$

and that of the surface $= (t + x) \times \frac{u}{2}$,

differentiating, therefore, supposing the radical constant, we find

$$u\, d u + (x - t)\,(d x - d t) = 0,$$
$$d t + d u + d x = 0,$$
$$u\,(d t + d x) + (t + x)\, d u = 0.$$

The second gives $d t = -d x - d u$,

and substituting this value in the two others,

$$u\, d u + (x - t)\,(2\, d x + d u) = 0,$$

and $-u\, d u + (t + x)\, d u = 0.$

This last gives $d u = 0$; whence the preceding becomes

$(x - t)\, 2\, d x = 0$, whence $x = t$.

This being determined, the equations of the perimeter and surface are reduced to $2t+2u=a$, and surface $=tu$.

The lines AB. AD, DC, CB, are all equal; and since the angle A must be a right angle, the other angles being such, the quadrilateral sought is a square.

We might have arrived at this property more readily, but that was not our principal object. We wished to show how the liberty of treating certain quantities as constant, may, in many cases, much facilitate the operation, and this example was well adapted to the purpose, as, without this artifice, the calculation would have been very complicated; similar methods may be applied to other polygons, and it will be found, that, in general, of all the polygons of the same number of sides, that is greatest which is a regular polygon. Whence it follows, that, of all figures of the same perimeter, the circle is that which contains the greatest space.

Of Multiple Points.

62. We have examined what takes place when one of the two differentials dx or dy, or which is the same thing, when the numerator or denominator of the fraction $\frac{dx}{dy}$ becomes zero; and we have found that one of these two cases always *exists* whenever there is a *maximum* or *minimum*. But it may be asked what takes place, if the denominator and numerator of the value $\frac{dx}{dy}$ become zero at the same time, and to what, in that case, is the value of $\frac{dx}{dy}$ reduced?

In answer to these questions, we first observe, that when we differentiate the equation of a curve, as there are only terms multiplied by dx and those multiplied by dy, we may, calling A the sum of the former, and B the sum of the latter, represent the differential equation by $A\,dx+B\,dy=0$. This equation gives $\frac{dx}{dy}=-\frac{B}{A}$; now, in order that A and B should become zero at the same time, they must have a common divisor, which, becoming zero when x and y have certain values, renders B and A equal at the same time to zero. For example, in the curve which has for its equation $y^2=\frac{x(a-x)^2}{a}$,

we have
$$\frac{dx}{dy}=\frac{2ay}{(a-x)^2-2x(a-x)},$$
or, substituting for y, its value,
$$\frac{dx}{dy}=\pm\frac{2a(a-x)\sqrt{\frac{x}{a}}}{(a-x)^2-2x(a-x)},$$
a quantity which becomes $\frac{0}{0}$, when $x=a$; but we see, at the same time, that $a-x$ is a common divisor of the numerator and the denominator, and that the value of $\frac{dx}{dy}$ is reduced to

$$\frac{dx}{dy}=\pm\frac{2a\sqrt{\frac{x}{a}}}{a-3x},$$

which, in the case of $x=a$, is reduced to ∓ 1, that is, in this example, the value of $\frac{0}{0}=\mp 1$.

We may indeed proceed thus; but this expedient is not always sufficient when the value of $\frac{dx}{dy}$ contains more than one variable, nor when it contains radical quantities, even if it have but one variable. It is therefore necessary to give an easier and more general method. But it is first necessary to show the nature of the points of curved lines where this singular case occurs. It takes place, as we shall soon see, at *multiple points*, that is, at those points where several branches of the same curve meet.

63. Let us conceive that $SOMZM'ON$ (*fig.* 20) is a curve of which two branches, at least, intersect at the point O. It is evident that to each value of AP, or x, within a certain interval, there correspond several values of y, as PM, PM', and those belonging to the branches which intersect each other, become equal at the point of intersection O.

In like manner, AZ being the axis of ordinates, to each ordinate AQ within a certain extent, there must correspond several abscissas QN, QN', QN''; and those belonging to the branches which intersect each other, must become equal at the point of intersection.

If, therefore, we represent by a the value of x, and by b that of y, which belong to the multiple point, the equation of the curve must be such, that when we substitute in it a for x, we shall find for y as many values equal to b as there are branches passing through the multiple point; and when we substitute b for y, we shall find a like number of values for x equal to a.

It follows from this that the equation must be such as to admit of being reduced to this form,

$$(x-a)^m F+(x-a)^{m-1}(y-b)F'+(x-a)^{m-2}(y-b)^2F''+(x-a)^{m-3}(y-b)^3F'''$$
$$+\cdots\cdots\cdots\cdot+T(y-b)^m=0.$$

m indicating the degree of multiplicity of the point in question, and F, F' &c., T, designating quantities composed in any way of x, y, and constants, or, as they are called for conciseness' sake, *functions* of x, y, and constants.

Indeed, it is evident, that if we make $x=a$, the equation which is then reduced to $(y-b)^m\,T=0$, will be divisible, m times, by $y-b$, and will consequently give m times the equation $y-b=0$, or $y=b$. So, if we make $y=b$, the equation which is then reduced to $(x-a)^m\,F=0$, will be divisible m times by $x-a$, and will consequently give so many times the equation $x-a=0$, or $x=a$; which cannot happen unless the equation is reducible to the form above.

Let us now conceive this equation to be differentiated m times in succession, making also dx and dy to vary, that it may be the more general. If we reflect on the principle of the differentiation, we shall readily perceive (as will presently be demonstrated by an example),

1st, that the last differential equation will be the only one, in which there are any terms not affected by $y-b$, or $x-a$. Therefore, whenever there is a multiple point, the first, second, third, &c. differentials of the equation, must, when instead of x and y, are substituted their values a and b answering to the multiple point, be all made to disappear, except those, the degree of whose differential is marked by m; 2d. It will also be perceived that, in this last differential equation, the terms affected by ddx, ddy, d^3x, &c., and by all differentials of higher degrees, will each have as factor some power of $x-a$, or of $y-b$; and consequently, that these differentials will disappear at the multiple point.

From these principles it follows, 1st. That at the multiple point we cannot have the value of $\frac{dx}{dy}$ expressed otherwise than by $\frac{0}{0}$, unless by the last differential equation, since all the other differential equations being then rendered equal to nothing, the factor of dx, as well as that of dy, becomes equal to zero. 2d. That as this last differential equation contains neither ddx, ddy, nor any higher differential, it might be derived immediately from the differentiation of the proposed equation, m times successively, supposing dx and dy constant. 3d. That in this last differential equation, dx and dy will be marked by the degree m; and, that consequently if we divide by dy^m, we shall have, by resolving the equation, m values of $\frac{dx}{dy}$, which will serve to find the tangents, which the different branches passing through the multiple point, have at that point.

To illustrate and confirm this by an example, let us take the curve which has for its equation

$$a(y-b)^2-x(x-a)^2=0.$$

If we differentiate this equation m times, that is, in this case, twice, we shall have, first,

$$2ady(y-b)-dx(x-a)^2-2xdx(x-a)=0;$$

secondly,

$$2addy(y-b)+2ady^2-ddx(x-a)^2-2dx^2(x-a)$$
$$-2xddx(x-a)-2dx^2(x-a)-2xdx^2=0.$$

In which we see that if we substitute a for x, and b for y, the first differential equation disappears, and in the second, the terms affected by ddx and ddy become nothing, so that it is reduced to

$$2ady^2-2adx^2=0.$$

But if, instead of considering dx and dy as variable, in the second differentiation, we had considered them as constant, we should have had,

$$2ady^2-2dx^2(x-a)-2dx^2(x-a)-2xdx^2=0,$$

which, on substituting a for x, is also reduced to $2ady^2-2adx^2=0$, and gives $\frac{dx}{dy}=\pm 1$, a result which indicates that there are two tangents at the point where $x=a$, and $y=b$; this point is therefore a double point, and the value $\frac{ydx}{dy}$ of the subtangent, becoming

then $= \pm b$, these two tangents make an angle of 45° with the ordinate. This will be confirmed by the description of the curve by means of its equation, which, giving

$$y = b \pm (x - a) \sqrt{\frac{x}{a}},$$

shows that the curve has two branches perfectly equal and similar, which interest each other at the point O, where $x = a$, and $y = b$. Its figure is such as is represented (*fig.* 20).

64. It is easy to determine by these principles whether a curve, of which we know the equation, have multiple points or not, what they are and where. We must differentiate the equation; put equal to zero the multiplier of dx and that of dy. These two equations will determine the value or values of x and y, according as there are one or more multiple points; but to be assured of the existence of this multiple point, we must examine whether these values of x and y satisfy the proposed equation. Then, to ascertain the degree of multiplicity of the point or points found, we differentiate the equation anew, but, for the sake of greater simplicity, considering dx and dy as constant. If, when the values found for x and y are substituted in this second differential equation, all the terms do not disappear, then the point found is only double. In the contrary case it is more than double. We proceed therefore to a third differentiation, still considering dx and dy as constant: and having substituted the values of x and y, the point will be triple if all the terms do not disappear, otherwise it will be at least quadruple. We continue to differentiate and substitute, until we arrive at a differential, of which all the terms are not made to disappear by the substitution of the values of x and y.

For example, if it is required to find the multiple points of the curve, which has for its equation

$$y^4 - axy^2 + bx^3 = 0,$$

we differentiate this equation, and obtain

$$4y^3\,dy - 2axy\,dy - ay^2\,dx + 3bx^2\,dx = 0.$$

Making the coefficient of dx and that of dy equal to zero, we have

$$4y^3 - 2axy = 0, \text{ and } 3bx^2 - ay^2 = 0.$$

The first of these two equations gives

$$y = 0, \text{ or } 4y^2 - 2ax = 0.$$

The value of $y = 0$, being substituted in the equation

$$3bx^2 - ay^2 = 0, \text{ gives } 3bx^2 = 0, \text{ or } x = 0;$$

now the proposed equation is satisfied by substituting 0 for x and y; therefore the curve has a multiple point corresponding to $x = 0$ and $y = 0$, that is, at the origin.

As to the value $4y^2 - 2ax = 0$, or $y^2 = \frac{ax}{2}$, if we substitute it in $3bx^2 - ay^2 = 0$, we have $3bx^2 - \frac{a^2x}{2} = 0$, whence we deduce $x = 0$, or $x = \frac{a^2}{6b}$, but the first, viz. $x = 0$, gives $y = 0$, which

indicates the same point as before; from the second we deduce $y^2 = \frac{a^3}{12b}$; but these values of x and y^2 do not satisfy the equation proposed. There is therefore no other multiple point than that found at the origin.

To ascertain its multiplicity, we differentiate a second time; we have

$$12y^2 dy^2 - 2axdy^2 - 2aydxdy - 2aydxdy + 6bxdx^2 = 0,$$

of which all the terms disappear on substituting for x and y their values, zero. Therefore the point is more than double.

We pass then to a third differentiation; we find

$$24ydy^3 - 2adxdy^2 - 2adxdy^2 - 2adxdy^2 + 6bdx^3 = 0;$$

or, substituting for x and y their values, zero,

$$6bdx^3 - 6adxdy^2 = 0;$$

as this third differential does not disappear, the point in question is a triple point.

To determine its tangents, we divide this equation by $6b$ and dy^3, and have

$$\frac{dx^3}{dy^3} - \frac{adx}{bdy} = 0,$$

which gives

$$\frac{dx}{dy} = 0, \text{ and } \frac{dx^2}{dy^2} - \frac{a}{b} = 0, \text{ or } \frac{dx}{dy} = \pm\sqrt{\frac{a}{b}}.$$

The first value, $\frac{dx}{dy} = 0$, indicates that one of the tangents to the multiple point is parallel to the ordinates, that is, that one of the branches touches the axis of the ordinates; since the multiple point is also at the origin. The two values

$$\frac{dx}{dy} = \pm\sqrt{\frac{a}{b}},$$

show that the two other branches make with the axis of the ordinates each an angle of which the tangent is $\sqrt{\frac{a}{b}}$, and that they extend on different sides of that axis. We may know the figure of this curve by resolving its equation, which gives

$$y = \pm\sqrt{\frac{ax}{2} \pm \frac{x}{2}\sqrt{a^2 - 4bx}},$$

taking for a and b two numbers at pleasure, and successively giving to x several values both positive and negative; it will be found to be such as is represented in figure 21.

Finally, when we have determined a multiple point by the operations given above, we must not always conclude that all the branches, which are considered as passing through that point, are visible. It may happen that the equation which furnishes the tangents, has imaginary roots; and then there are so many invisible branches. The points where this happens, are sometimes detached from the course of the curve to which they nevertheless belong; they are then called *con-*

jugate points. But whether detached or not, they are not the less considered as having the number of branches indicated by the degree of their multiplicity: the curve to which they belong is an individual of a more extensive family, in which all these branches are visible; but they become invisible in this, because some one of the constant quantities which enter into the equation common to the whole family, becomes zero in the particular case of this individual curve. It is thus that in the curve, which has for its equation

$$m\,(y-b)^2 - x\,(x-a)^2 = 0,$$

(and of which the curve in figure 20 is a particular case, and which it becomes when $m = a$) the *leaf*, which this curve has, no longer exists when a is supposed $= 0$, which reduces the equation to

$$m\,(y-b)^2 - x^3 = 0.$$

The two branches OMZ, $OM'Z$, which were above the point O, will no longer exist in this last, or at least will not be visible; for we may suppose that they are still there, regarding a not as absolutely zero, but only as infinitely small. There will nevertheless be two tangents, in fact; but this example enables us to conceive how certain branches may disappear.

65. Several useful consequences may be drawn from what has just been said on the subject of multiple points.

66. 1st. When a fractional algebraical expression, into which there enter one or two variables, is such, that by substituting in it for each of those variables, certain determinate values, it becomes $\frac{0}{0}$, we shall obtain the value which this expression ought then to have, by differentiating separately the numerator and the denominator, as many times in succession as is necessary, in order that they may not become zero at the same time; and in this differentiation we may treat the first differences as constants. Indeed we may always consider any fractional algebraical expression $\frac{B}{A}$ containing two variables, for instance, as being the value of $\frac{dx}{dy}$ (x and y being these two variables); that is, we may always suppose $\frac{dx}{dy} = \frac{B}{A}$, and consequently $A\,dx - B\,dy = 0$. But since, by the supposition, A and B become zero at the same time, when x and y have certain values, it follows from what precedes that to obtain the value of $\frac{dx}{dy}$, we must differentiate this equation, considering dx and dy as constant, until we arrive at an equation which does not disappear by the substitution of the values of x and y. Now these successive differentiations give

$$\begin{aligned} dA\,dx - dB\,dy &= 0, \\ ddA\,dx - ddB\,dy &= 0, \\ d^3A\,dx - d^3B\,dy &= 0, \end{aligned}$$

whence we deduce

$$\frac{dx}{dy} = \frac{dB}{dA},\ \frac{dx}{dy} = \frac{ddB}{ddA},\ \frac{dx}{dy} = \frac{d^3B}{d^3A};$$

that is, we must differentiate separately the numerator and denominator as has been directed; and the last of these equations will give the value of $\frac{dx}{dy}$.

67. 2d. When an equation, which contains several variables, is such, that to certain values of all those variables except one, there shall correspond a certain number of equal values of that one, and this shall hold true of all the rest, we shall find these values by differentiating successively so many times less one, the equation proposed; considering, in these differentiations, dx, dy, dz, &c. constant, if x, y, z, &c. are the variables; and putting equal to zero the multipliers of dx, dy, dz, &c., those of $dx^2, dx\,dy, dz\,dy$, &c., and so throughout. If all these equations correspond with each other and with the equation proposed, the values of x, y, z, &c., which they give, will be the values required.

68. It may be observed with regard to multiple points, that since the values of x and y, found by putting equal to zero the coefficient of dx and dy, must satisfy the finite equation proposed, we cannot, unless the equation is given in finite terms, or may be reduced to them by the methods of the integral calculus, ascertain, by calculation alone, the existence of those points.

Of the visible and invisible points of inflexion.†

69. It sometimes happens, that a branch of a curve, after having been concave towards a certain point, becomes afterwards, in its progress, convex towards it (without discontinuing its course). Such is the curve represented in figure 22. The point O, where this change takes place, is called the *point* of *inflexion.*

To determine these points it must be observed that the tangent to the point O must be a common tangent of the branch OB and the branch Ob, which meet at the point O; if, therefore, on each side of O, we take two arcs, equal or unequal, but infinitely small, the tangent must be the prolongation of each arc, so that the two little arcs must be in the same straight line.

This being laid down, we draw the ordinates MP, OQ, mp, and call AP, x, and PM, y. We then have $Mr = dx$, $Or = dy$, and supposing OM and Om to differ infinitely little from each other, we have

$$Or' = d(x + dx) = dx + ddx, \text{ and } mr' = dy + ddy;$$

since then MO and Om are in a straight line, the triangles MrO and $Or'm$ are similar: and if we suppose, as we are at liberty to do, that the arcs MO and Om are equal, these triangles will be also equal, and we shall have $Mr = Or'$, and $Or = mr'$; that is,

$$dx = dx + ddx, \text{ and } dy, = dy + ddy,$$

therefore

$$ddx = 0, \text{ and } ddy = 0.$$

† These are sometimes called points of contrary flexure.

In order, therefore, to determine the simple points of inflexion, we must differentiate twice successively the equation of the curve, and consider in the second differential equation, $d\,d\,x = 0$, and $d\,d\,y = 0$. Now it is evident that, in that case, this second equation is the same as if we had differentiated, considering $d\,x$ and $d\,y$ constant. If therefore, from the first differential equation we deduce the value of $d\,x$ or $d\,y$ and substitute it in the second, we shall have an equation, which, being divided by $d\,y^2$ or $d\,x^2$, will contain only x, y and constants, and which, being compared with the equation of the curve, will give the values of x and y, which correspond to the simple point of inflexion.

Let us take, as an example, the curve which has for its equation

$$x^3 - b\,y^2 = a^3.$$

We shall have $$3\,x^2\,d\,x - 2\,b\,y\,d\,y = 0,$$

differentiating again, treating $d\,x$ and $d\,y$ as constant, we have

$$6\,x\,d\,x^2 - 2\,b\,d\,y^2 = 0;$$

the first of these gives

$$d\,y = \frac{3\,x^2\,d\,x}{2\,b\,y};$$

substituting this value in the second, we have

$$6\,x\,d\,x^2 - \frac{18\,b\,x^4\,d\,x^2}{4\,b^2\,y^2} = 0,$$

or $$4\,b\,y^2 - 3\,x^3 = 0.$$

From this we deduce $y^2 = \dfrac{3\,x^3}{4\,b}$, which, being substituted in the equation of the curve, gives

$$x^3 - \tfrac{3}{4}\,x^3 = a^3,\ \text{or}\ x^3 = 4\,a^3,$$

therefore $$x = a\sqrt[3]{4},$$

and consequently

$$y = \sqrt{\frac{3\,a^3}{b}} = a\sqrt{\frac{3\,a}{b}}.$$

These are the values which determine the point of inflexion.

Let us take, as a second example, the curve which has for its equation $$y = a + (x - a)^{\frac{3}{5}},$$

we have $$d\,y = \tfrac{3}{5}\,(x - a)^{-\frac{2}{5}}\,d\,x,$$

differentiating anew, considering $d\,y$ and $d\,x$ as constant, we find

$$-\tfrac{6}{25}\,(x - a)^{-\frac{7}{5}}\,d\,x^2 = \frac{-\frac{6}{25}\,d\,x^2}{(x - a)^{\frac{7}{5}}} = 0;$$

therefore $d\,x = 0$; now the first differential equation

$$d\,y = \tfrac{3}{5}\,(x - a)^{-\frac{2}{5}}\,d\,x,$$

becomes

$$d\,y = \frac{\frac{3}{5}\,d\,x}{(x - a)^{\frac{2}{5}}},\ \text{or}\ (x - a)^{\frac{2}{5}}\,d\,y = \tfrac{3}{5}\,d\,x;$$

but since $dx = 0$, we have $(x-a)^{\frac{2}{5}} dy = 0$, which gives either $dy = 0$, or $(x-a)^{\frac{2}{5}} = 0$; but as it is not possible that we should have, at the same time dx and $dy = 0$, it follows that $(x-a)^{\frac{2}{5}} = 0$ is the true solution, which gives $x = a$, and consequently $y = a$.

70. We may here observe, 1st, that as we find $dx = 0$, the tangent at the point of inflexion of this curve is parallel to the ordinates.

2d. If the curve should have several points of inflexion, the final equation would give several values of x, that is, it would exceed the first degree. This takes place in curves which have a serpentine course, as in figure 23.

71. If we conceive that the two points of inflexion O and O' (*fig.* 23) approach each other continually, and are at last infinitely near each other; then if we represent, as above, the two infinitely small arcs OM and Om, and the two other infinitely small arcs $O'M'$ and $O'm'$, on each side of the points of inflexion O and O', the two sides Om and $M'O'$ will be, or may be supposed to be, one on the other; and since, at the point of inflexion, MO is in a straight line with Om, and $M'O'$ with $O'm'$, there will then be three small consecutive arcs in a straight line.

This being laid down, let Mm, mm', $m'm''$ (*fig.* 24) be these three infinitely small arcs. Let fall the ordinates MP, mp, $m'p'$, $m''p''$, and draw the lines Mr, mr', $m'r''$, parallel to AP. Call AP, x, and PM, y. We have

$$Mr = dx, rm = dy, mr' = dx + ddx, r'm' = dy + ddy,$$
$$m'r'' = dx + ddx + d^3x, r''m'' = dy + ddy + d^3y.$$

Now the three triangles Mrm, $mr'm'$, $m'r''m''$ are similar, since the sides Mm, mm', $m'm''$ are in a straight line; if therefore we suppose these sides equal, which we are at liberty to do, the triangles will be equal. We shall have then

$$dx = dx + ddx, dy = dy + ddy, dx + ddx = dx + ddx + d^3x,$$
$$dy + ddy = dy + ddy + d^3y;$$

that is,

$$ddx = 0, ddy = 0, d^3x = 0, d^3y = 0.$$

If, therefore, we differentiate the equation of the curve three times successively, considering as variable dx, dy, ddx, ddy, and afterwards put 0 for each of the quantities ddx, ddy, d^3x, d^3y, each of the three equations resulting from those differentiations will hold true. Now it is evident that they then become the same as if we had differentiated three times in succession, supposing dx and dy constant.

By a similar course of reasoning it may be shown that if three consecutive points of inflexion come to unite, there will be, at the point of union, four elements in a straight line, and it may be proved in the same way, that if we differentiate the equation four times in succession, supposing dx and dy constant, the four resulting equations will *hold true;* and so of others.

Therefore, if, from the first differential equation, we deduce the value of dx, and substitute it in all the others, we shall have, after

dividing the second by dy^2, the third by dy^3, and so forth, so many equations, which must *hold true* conjointly with the proposed equation, in order that there may be one, two, three, &c. inflexions united. If, therefore, from two of these equations we deduce the values of x and y, these values, substituted in the other equations, must satisfy them.

When there are only two inflexions united, the inflexion is invisible, it becomes visible when there are three; in general, the inflexion is visible or invisible, according as the number of inflexions united is odd or even. Therefore if $E = 0$, represent generally the equation of a curve, it will be necessary, in order that there should be m points of inflexion united, that, differentiating E, $m + 1$ times, on the supposition that dx and dy are constant, all the differentials ddx, ddy, d^3x, d^3y, &c. to that of the degree $m + 1$, inclusively, should be zero; and the inflexion will be visible or invisible according as m is odd or even.

72. Hitherto the ordinates have been supposed parallel. If they issue from a fixed point, the following is the method of determining the points of inflexion. Imagining the two infinitely small consecutive arcs, which, at the point of inflexion, must be in a straight line, we draw the ordinates (*fig.* 25) CM, Cm, Cm', and describe the arcs Mr, mr', which may be regarded as perpendicular to Cm and Cm'. This done, it is easy to see that the angle $r'mm'$ differs from the angle rMm by the angle mCr'. For we have $Cmm' + rmM = 180°$, or

$$Cmr' + r'mm' + 90° - rMm = 180°;$$

therefore $$r'mm' - rMm = 90° - Cmr';$$

but by the triangle $Cr'm$ right-angled at r', we have

$$90° = Cmr' + mCr';$$

therefore $$r'mm' - rMm = mCr'.$$

If we draw the line mn, making the angle $m'mn = mCr'$, the angle nmr' will be equal to mMr, and consequently the triangle tmr' will be similar to the triangle mMr. Calling CM, y, and Mr, dx, we have

$$mr = dy,\ mr' = dx + ddx, \text{ and } m'r' = dy + ddy;$$

call Mm, ds, mm' will be $= ds + dds$. Describe from the point m with a radius $= mm'$, the arc $m'n$: the sectors Cmr' and nmm' will be similar, and will give

$$Cm : mr' :: mm' : m'n;$$

that is, $$y + dy : dx + ddx :: ds + dds : m'n,$$

which, omitting the quantities which may be neglected, gives

$$m'n = \frac{ds\,dx}{y};$$

but the triangle $m'tn$, which is similar to $tr'm$, will be so likewise to mrM; we have, therefore, $Mr : Mm :: m'n : m't$; that is,

$$dx : ds :: \frac{ds\,dx}{y} : m't = \frac{ds^2}{y};$$

therefore $$r't = dy + ddy - \frac{ds^2}{y}.$$

Now the similar triangles $M\,r\,m$ and $m\,r'\,t$ give

$$M\,r : r\,m :: m\,r' : r'\,t,$$

or $$d\,x : d\,y :: d\,x + d\,d\,x : d\,y + d\,d\,y - \frac{d\,s^2}{y};$$

therefore

$$d\,x\,d\,y + d\,x\,d\,d\,y - \frac{d\,x\,d\,s^2}{y} = d\,x\,d\,y + d\,y\,d\,d\,x,$$

or $$d\,x\,d\,d\,y - d\,y\,d\,d\,x - \frac{d\,x\,d\,s^2}{y} = 0;$$

this is the formula for finding the simple points of inflexion, when the ordinates issue from a fixed point; it becomes that for the parallel ordinates, when we suppose the point C at an infinite distance, and $d\,x$ and $d\,y$ constant; for then the term $\frac{d\,x\,d\,s^2}{y}$ must be rejected, as y is infinite. In the application of this formula, it will always be more simple to suppose $d\,x$ constant, which reduces it to

$$d\,d\,y = \frac{d\,s^2}{y};$$

and care must be taken to treat $d\,x$ as constant in the differentiation of the equation of the curve: but as the lines $d\,x$ are arcs described with a variable radius, while, if the ordinates issue from a fixed point, they are referred to arcs described with a constant radius CO, as has been observed (39), we must be careful to substitute for $d\,x$ its value, which will always be easily found, by observing that the sectors $CS\,s$ and $CM\,r$ are similar. It is not necessary to give the formulas for the other points of inflexion, on the same supposition, although it would not be difficult.

Observations on the Maxima and Minima.

73. Let F be a function of one or more variables, and let it be susceptible of a *maximum* or *minimum*. If we imagine it to represent the ordinate or abscissa x of a curve, that is, if we suppose $x = F$, it follows from what has been said on the subject of the points of inflexion, that if it be true that $d\,F$ or $d\,x$ becomes zero, whenever there is a *maximum* or a *minimum*, the converse is not always true; for we have seen a case (69) where $d\,x$ was found $= 0$ at the point of inflexion, that is, the tangent became parallel to the ordinates; but it is evident that neither the ordinate nor the abscissa is then either a *maximum* or *minimum*. What then must be done to ascertain the existence of a *maximum* or *minimum?* We must differentiate the quantity several times in succession, considering the first differentials of each variable as constant; and, if the values which the variables have at the point of the *maximum* or *minimum* sought, cause $d\,F$, and $d\,d\,F$ to disappear, but not $d^3\,F$, there is no *maximum* in the curve which has for its equation $x = F$; but there is a visible point of inflexion; so that the quantity F has no *maximum* or *minimum*. But if $d^3\,F$ disappears, and not $d^4\,F$, there will be a

maximum or *minimum*. In general, there will be a *maximum* or *minimum*, if the last differential made to disappear is of a degree marked by an odd number.

Of Cusps of different species, and of the different sorts of contact of the branches of the same curve.

74. When two branches of a curve come in contact, they may either have their convexities opposed to each other, as in *figure* 26, or have their concavities opposed, and thus one embrace the other, as in *figure* 27, and in both cases it may either happen that they continue their course on each side of the point of contact, or that they stop suddenly at that point, as we see in *figures* 28 and 29. In this last case, the point of contact is called a *cusp;* that represented in *figure* 28 is called a *simple cusp* or *cusp of the first species;* that in *figure* 29 is called a *beaked cusp* or *cusp of the second species.* If more than two branches unite, their different varieties may be found at once at the same point, and there may also be found an infinite number of others; for example, the branches may, at the point of contact, undergo one or more inflexions. It may happen that an inflexion and a cusp occur united, so as to seem to form only a single cusp. In *figure* 30, if the branch EBD, which forms at E, with the branch EC, a cusp of the first species, had a point of inflexion at B, and if the point of inflexion B should be infinitely near the point E, we should only be presented with the appearance of a cusp of the second species. These varieties may be infinite, especially if we consider that several branches may touch at once. We shall not undertake to give the character of each; we shall only observe, that whenever several branches of a curve touch each other, it may be ascertained by the following *facts.* 1st. This point being multiple must have the conditions enumerated in *art.* 63. 2d. The equation which serves to determine the tangents of the multiple points, must then have as many equal roots as there are branches which touch, since there must be so many tangents united. Thus, for cusps of the first species, there ought to be the conditions common to double points, and the equation which gives the tangents thereof, or which gives $\frac{dx}{dy}$, ought to have two equal roots. As the consideration of these points is not sufficiently useful to engage us in the details which their investigation would require, the subject will be no farther pursued.

On the Radii of Curvature and the development or evolute.

75. If upon each of the points M, m, m', &c. of any curve line (*fig.* 31), we conceive the perpendiculars MN, $m\,n$, $m'n'$, &c. to be raised; the consecutive intersections N, n, n', will form a curve line, to which has been given the name of evolute, because if we consider it as *enveloped* by a thread ABN, which touches it in its origin B, then, upon unwinding the thread, the extremity A traces the curve AM.

In fact, in the development of $N n$, for example, considering $N n$ as a small straight line, the thread MNn describes, about the point n, as a centre, the arc $M m$, to which it is necessarily perpendicular, since the radius of a circle is perpendicular to its circumference.

76. The curve AM being given, if we wish, for any point M, to determine the value of $M n$, which is called the radius of curvature, we observe that $M n$ is determined by the concourse of two perpendiculars infinitely near each other, MN and $m n$. We therefore imagine (*fig.* 32) two consecutive arcs $M m, m m'$, infinitely near, and differing infinitely little from each other, which may be considered as two straight lines; we also imagine MN perpendicular to $M m$ at M, and $m N$ perpendicular to $m m'$ at m. Then, in the triangle $NM m$, right-angled at M, we shall have

$$1 : \sin MN m : : m N \text{ or } MN : M m,$$

or, because the angle $MN m$ is infinitely small,

$$1 : MN m : : MN : M m \text{; therefore } MN = \frac{M m}{MN m};$$

but, if we produce Mm, we have $N m u = NM m + MN m$; because $N m u = N m m' + m' m u = NM m + MN m$, and taking away $N m m' = NM m$, there remains $m' m u = MN m$; therefore

$$MN = \frac{m M}{m' m u}.$$

If we draw Mr and $m r'$ parallel to AP, it is easy to see that the angle $u m r'$ being equal to $m M r$, the angle $u m m'$ is the quantity by which the angle $m M r$ is diminished, or the differential of the angle $m M r$, which must be taken negatively here, where the curve is concave, and where it is convex, must have the sign plus; we have therefore $MN = \dfrac{M m}{\mp d (r M m)}$. We have therefore to find the expression for $d (r M m)$. Now the tangent of $r M m = \dfrac{d y}{d x}$, and

$$\cos r M m = \frac{dx}{d s},$$

$d s$ designating the arc $M m$; but we have seen (23), that z being any arc, we had $d z = \cos z^2 \, d (\text{tang } z)$; therefore

$$d (r M m) = \frac{d x^2}{d s^2} d \left(\frac{d y}{d x}\right);$$

whence,

$$MN = \mp \frac{d s}{\dfrac{d x^2}{d s^2} d \left(\dfrac{dy}{d x}\right)} = \frac{d s^3}{\mp d x^2 d \left(\dfrac{d y}{d x}\right)}.$$

This is the formula for finding the radius of curvature when the ordinates are parallel.

77. If the ordinates are supposed to issue from a fixed point (*fig.* 33), we have, as above, and for the same reasons,

$$MN = \frac{M m}{m' m u},$$

but $m'mu$ is no longer $= \mp d(rMm)$, but the value of this angle is found as follows.

Describing the arcs Mr, mr', we have

$$m'mu = r'mu - r'mm';$$

but it was shown (72) that $r'mu$ differs from rMm by the angular quantity mCr' or MCr, for the two last quantities differ from each other only by a quantity infinitely small compared with them; therefore $r'mu = MCr + rMm$; therefore, since

$$m'mu = r'mu - r'mm',$$

we have

$$m'mu = MCr + rMm - r'mm' = MCr - d(rMm).$$

Now, calling Mr, dx, we have

$$1 : \sin MCr \text{ or } : MCr :: CM : Mr;$$

that is, $1 : MCr :: y : dx$; therefore $MCr = \frac{dx}{y}$, and since

$$d(mMr) = \frac{dx^2}{ds^2}\, d\left(\frac{dy}{dx}\right),$$

we have $$m'mu = \frac{dx}{y} - \frac{dx^2}{ds^2}\, d\left(\frac{dy}{dx}\right);$$

and if the curve were convex, we should find, in like manner,

$$m'mu = \frac{dx}{y} + \frac{dx^2}{ds^2}\, d\left(\frac{dy}{dx}\right);$$

therefore

$$MN = \frac{ds}{\frac{dx}{y} \mp \frac{dx^2}{ds^2}\, d\left(\frac{dy}{dx}\right)} = \frac{y\,ds^3}{ds^2\,dx \mp y\,dx^2\, d\left(\frac{dy}{dx}\right)}.$$

78. To give an application of these formulas, let us suppose that the curve AM (*fig.* 32) is a circle, having for its equation

$$y^2 = 2ax - x^2,$$

we shall have $y = \sqrt{2ax - x^2}$; therefore

$$dy = \frac{a\,dx - x\,dx}{\sqrt{2ax - x^2}},$$

and consequently,

$$ds = \sqrt{dx^2 + dy^2} = \frac{a\,dx}{\sqrt{2ax - x^2}}; \quad (29)$$

and $\frac{dy}{dx} = \frac{a - x}{\sqrt{2ax - x^2}}$; therefore $d\left(\frac{dy}{dx}\right) = \frac{-a^2\,dx}{(2ax - x^2)^{\frac{3}{2}}}$;

the formula which belongs to the present case, in which the curve is concave and the ordinates parallel, is $\frac{ds^3}{-dx^2\, d\left(\frac{dy}{dx}\right)}$,

which is changed into

$$\frac{\dfrac{a^3\,dx^3}{\left(2\,a\,x-x^2\right)^{\frac{3}{2}}}}{\dfrac{a^2\,dx^3}{\left(2\,a\,x-x^2\right)^{\frac{3}{2}}}}=a$$

that is, the radius of curvature is always of the same magnitude and equal to the radius of the circle; so that the evolute is reduced to a point, which is the centre; and this agrees with what we know of the circle.

Let us take, as a second example, the parabola which has for its equation $y^2=a\,x$, or $y=\sqrt{a\,x}=a^{\frac{1}{2}}\,x^{\frac{1}{2}}$, we shall have

$$d\,y=\tfrac{1}{2}\,a^{\frac{1}{2}}\,x^{-\frac{1}{2}}\,d\,x,$$

therefore

$$d\,s=\sqrt{d\,x^2+d\,y^2}=\sqrt{d\,x^2+\tfrac{1}{4}\,a\,x^{-1}\,d\,x^2}=d\,x\,\sqrt{1+\frac{a}{4\,x}}$$

$$=d\,x\,\sqrt{\frac{4\,x+a}{4\,x}}=\tfrac{1}{2}\,x^{-\frac{1}{2}}\,d\,x\,\sqrt{4\,x+a},\text{ and }\frac{d\,y}{d\,x}=\tfrac{1}{2}\,a^{\frac{1}{2}}\,x^{-\frac{1}{2}};$$

therefore $\quad d\left(\dfrac{d\,y}{d\,x}\right)=-\tfrac{1}{4}\,a^{\frac{1}{2}}x^{-\frac{3}{2}}\,d\,x;$

the formula $\quad \dfrac{d\,s^3}{-\,d\,x^2\,d\left(\dfrac{d\,y}{d\,x}\right)}$

becomes then

$$\frac{\frac{1}{8}\,x^{-\frac{3}{2}}\,d\,x^3\,(4\,x+a)^{\frac{3}{2}}}{d\,x^2\times\frac{1}{4}\,a^{\frac{1}{2}}x^{-\frac{3}{2}}\,d\,x}=\frac{\frac{1}{2}\,(4\,x+a)^{\frac{3}{2}}}{a^{\frac{1}{2}}}=\frac{4\,x+a}{2}\sqrt{\frac{4\,x+a}{a}}.$$

79. The radii of curvature serve to measure the curvature of a curve at each point. Since, in the development of the element $n\,n'$ of the curve BN (*fig.* 31), the thread traces the small arc $m\,m'$, this arc has the same curvature as the circle which has for its radius the line $m\,n$. Thus when we have the expression for the radius of curvature, we know, for each point, the radius of the circle which has the same curvature as the curve has at that point. And as the curvature of a circle is greater in proportion as its radius is less, that is, as the curvatures of circles are in the inverse ratio of their radii, it will be easy to compare the curvature of a curve at any point with the curvature of the same or another curve at another point. Thus, if we wish to compare the curvature of the parabola at its origin, with that of the same curve when the ordinate passes through the focus, we observe that at the origin $x=0$, and that the abscissa corresponding to the focus is $\frac{1}{4}\,a$ (*Ap.* 172). Putting therefore, successively, for the expression of the radius of curvature, $x=0$, and $x=\frac{1}{4}\,a$, we have $\frac{1}{2}\,a$, and $a\sqrt{2}$; the radius of curvature is therefore $\frac{1}{2}\,a$ at the origin, and $a\sqrt{2}$ at the extremity of the ordinate which passes through the focus. Therefore the curvature at the first of these points is to the curvature at the second, as

$$a\sqrt{2}:\tfrac{1}{2}\,a,\text{ or}::2\sqrt{2}:1.$$

Since the radius of curvature MN is nothing but the thread which is supposed to have enveloped or been wrapped about the curve BN, it follows that it is equal in length to the arc BN, plus the part AB, by which the thread exceeded the curve when the development began, that is, plus the radius of curvature at the origin A. Therefore the curve BN is rectifiable, that is, we may assign the length of each of its arcs BN.

Remark.

80. The points of inflexion were determined on the supposition (69) that the two elements of the curve, near the point of inflexion, were in a straight line. From this supposition it seems to follow, that at the point of inflexion the radius of curvature must always be infinite, because the two perpendiculars upon the two consecutive sides must be parallel. There are, nevertheless, several curves which, at the point of inflexion, have the radius of curvature equal to zero; the parabola, for example, which has for its equation $y = x^{\frac{3}{5}}$.

But it must be observed that nothing in the supposition that was made, determines of what magnitude those two consecutive elements are. Now if they be each reduced to a point, they will be nevertheless in the same straight line, and the two perpendiculars falling one on the other, will meet at the very point from which they issue. And this is what happens in those curves where the radius of curvature is zero at the point of inflexion. For the curvature then being infinite, the two consecutive elements are each confounded with the tangent infinitely less than in any other case, and must consequently be considered as two points united. The two elements may therefore be in a straight line, without the radius of curvature being infinite; but we see from this, that the radius of curvature is, at the point of inflexion, either infinite or nothing.

ELEMENTS

OF THE

INTEGRAL CALCULUS.

Explanations.

81. THE method known by the name of the *Integral Calculus* is the reverse of the Differential Calculus. Its object is to ascend from differential quantities to the functions from which they are derived.

There is no variable quantity expressed algebraically of which the differential may not be found; but there are many differential quantities † which cannot be integrated; some, indeed, because they could not have resulted from any differentiation; such as the quantities $x\,d\,y$, $x\,d\,y - x\,d\,y$, &c., others, because means have not yet been discovered of integrating them, and among these last are some of which we may despair of ever finding the integral.

As, however, great use may be made of those which we know how to integrate, we shall endeavour to show the methods, and shall afterwards show what is to be done with reference to those which refuse to be integrated. We shall begin by explaining certain modes of expression which will hereafter be used.

We call a *function* of one or more quantities, any expression, into which those quantities enter in any way whatever, whether mixed or not with other quantities which are considered as having determinate and invariable values, while the quantities of the function may have all possible values. Thus, in a function, we consider only the quantities which are supposed variable, without any regard to the constants which may occur in it. For example, x, $a + b\,x^2$, $\sqrt[u]{a\,x^n + x^p}$, &c. are functions of x.

† By *Differential quantity* is meant here, not only such as results from a differentiation, but in general, every quantity affected by the differentials $d\,x$, $d\,y$, &c. of one or more variables.

By *algebraical quantities* are meant those, of which the exact value may be assigned, by executing a determinate number of algebraical or arithmetical operations, other than those which depend on logarithms. On the contrary, we call *non-algebraical* or *transcendental,* those quantities, for which we can assign only proximate values, or values by means of approximations; logarithms are of this kind, and an infinite number of others.

To indicate the integral of a differential, the letter $\int$ is written before this quantity; this letter is equivalent to the words *sum of,* because, *to integrate*, or *take the integral,* is nothing but to sum up all the infinitely small increments which the quantity must have received, to arrive at a determinate, finite state.*

Of the Differentials with a single variable, which have an algebraical integral; and first, of simple differentials.

82. Fundamental rule. *To integrate a simple differential, we must,* 1st, *increase the exponent of the variable by unity;* 2d, *divide by this exponent thus increased, and by the differential of the variable; that is, divide by this new exponent multiplied by the differential of the variable.*

The reason of this rule is evident from the principle of differentiation (10). As the object is to find the quantity which must have been differentiated, it is evident, that we must make use of operations the reverse of those employed for differentiating a quantity. This is made more clear by examples of the application of the rule.

$$\int 2\,x\,d\,x = \int 2\,x^1\,d\,x = \frac{2\,x^{1+1}\,d\,x}{(1+1)\,d\,x} = x^2\,;$$

$$\int x\,d\,x = \frac{x^2\,d\,x}{2\,d\,x} = \frac{x^2}{2}.$$

We see, now, that $d\,(x^2)$ is actually $2\,x\,d\,x$; and

$$d\left(\frac{x^2}{2}\right) = \frac{2\,x\,d\,x}{2} = x\,d\,x.$$

In like manner,

$$\int a\,x^{\frac{2}{3}}\,d\,x = \frac{a\,x^{\frac{2}{3}+1}\,d\,x}{(\frac{2}{3}+1)\,d\,x} = \frac{a\,x^{\frac{5}{3}}}{\frac{5}{3}} = \tfrac{3}{5}\,a\,x^{\frac{5}{3}}\,;$$

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* In reading, it is convenient to call $\int$, *the integral.* Thus $\int 2\,x\,d\,x$ is read, the *integral* of $2\,x\,d\,x$.

$$\int \frac{a\,d\,x}{x^3} = \int a\,x^{-3}\,d\,x = \frac{a\,x^{-3+1}\,d\,x}{(-3+1)\,d\,x} = \frac{a\,x^{-2}}{-2} = -\frac{a}{2\,x^2}.$$

In general, m being an exponent, positive, negative, **integer**, or fractional, we have

$$\int a\,x^m\,d\,x = \frac{a\,x^{m+1}\,d\,x}{(m+1)\,d\,x} = \frac{a\,x^{m+1}}{m+1}.$$

We have no need of this rule to find the integral of $d\,x$ or $a\,d\,x$, which, we see at once, must be x for the first and $a\,x$ for the second. But, if we wish to apply the rule to them, we observe that the exponent of x in these differentials is zero, and that they are the same thing as $x^0\,d\,x$ and $a\,x^0\,d\,x$, of which the integral, agreeably to the rule, is $\frac{x^{0+1}\,d\,x}{(0+1)\,d\,x}$ and $\frac{a\,x^{0+1}\,d\,x}{(0+1)\,d\,x}$, or x and $a\,x$.

There is only one case which eludes this fundamental rule; it is that in which the exponent m has the value -1; for then the integral becomes $\frac{a\,x^{-1+1}}{-1+1} = \frac{a\,x^0}{0} = \frac{a}{0}$, a quantity unassignable because infinite; for, if we conceive the denominator, instead of being zero, to be an infinitely small quantity, it must evidently be contained an infinite number of times in the finite quantity a, and consequently the fraction must be infinite. We shall explain, hereafter, the reason why the calculus gives in this case an infinite quantity; meanwhile we observe that the proposed differential, $a\,x^m\,d\,x$, which in this case is $a\,x^{-1}\,d\,x$, or $\frac{a\,d\,x}{x}$, is the differential of a logarithm. It is the differential of $a\,l\,x$ or of $l\,x^a$, as may be easily seen by differentiating (26).

If there were a radical in the simple differential, we should substitute, for the radical, a fractional exponent. Thus,

$$\int a\,d\,x\,\sqrt[3]{x^2} = \int a\,x^{\frac{2}{3}}\,d\,x = \tfrac{3}{5}\,a\,x^{\frac{5}{3}}.$$

Remark.

83. We have seen, that when, in the quantities to be differentiated, there occur terms wholly constant, these terms do not appear in the differential. When, therefore, we go back to the integral, we must take care to add a constant quantity to the result of the integration. This quantity will always have an indetermi-

nate value, as we have no other object than to find the integral, that is, to find a quantity, such that, by differentiating it, we reproduce the differential proposed; indeed,

$$\frac{a\,x^{m+1}}{m+1}, \text{ and } \frac{a\,x^{m+1}}{m+1} + C,$$

C being any constant quantity, have equally for their differential the quantity $a\,x^m\,d\,x$, whatever value is given to C. But when the integration is performed with a view of satisfying a given question, then this constant quantity has a value, determined by the state of the question. This will be shown hereafter; but, in future, care will be taken to add a constant quantity to the result of each integration; and that it may be known as such, it will be always designated by the letter C.

Of Complex Differentials whose integration depends on the fundamental rule.

84. 1st. We may integrate by the preceding rule, any quantity, in which there occur no powers of complex quantities, and no complex divisors, except such as are constant quantities.

Thus the entire integral of

$$a\,x^3\,d\,x + \frac{b\,x^2\,d\,x}{c} + e\,d\,x,$$

being the sum of the integrals of each of the terms, will be

$$\frac{a\,x^4}{4} + \frac{b\,x^3}{3\,c} + e\,x + C.$$

In like manner,

$$\int\left(a\,x^3\,d\,x + \frac{b\,d\,x}{x^4}\right) = \int\left(a\,x^3\,d\,x + b\,x^{-4}\,d\,x\right)$$

$$= \frac{a\,x^4}{4} + \frac{b\,x^{-3}}{-3} + C = \frac{a\,x^4}{4} - \frac{b}{3\,x^3} + C.$$

85. 2d. Even should there occur powers of complex quantities, they may still be integrated by the fundamental rule, provided they be not found in the denominator, and provided, also, their exponent be a positive whole number. For example, $(a + b\,x^2)^3\,d\,x$ may be integrated by the preceding rule, by actually raising $a + b\,x^2$ to the third power, which would give, (*Algebra,* 141), $a^3 + 3\,a^2\,b\,x^2 + 3\,a\,b^2\,x^4 + b^3\,x^6$, and consequently

$$\int(a+bx^2)^3\,dx=\int\left\{a^3dx+3a^2bx^2dx+3ab^2x^4\,dx+b^3x^6\,dx\right\}$$
$$=\int a^3\,dx+\int 3\,a^2\,bx^2\,dx+\int 3\,ab^2\,x^4\,dx+\int b^3\,x^6\,dx$$
$$=a^3\,x+\frac{3\,a^2\,b\,x^3}{3}+\frac{3\,a\,b^2\,x^5}{5}+\frac{b^3\,x^7}{7}+C.$$

86. As there is no complex quantity raised to a power indicated by a positive whole number, which may not, by the preceding rule (*Alg.* 141), be thus reduced to a finite series of simple quantities, we may always integrate any complex quantity which does not contain any other complex parts than powers whose exponent is a positive whole number. Thus, if we had to integrate

$$g\,x^3\,d\,x\,(a+bx^2)^2+a^2\,x^7\,d\,x\,(c+e\,x^2+f\,x^3)^4,$$

we should develope, by the rule already cited, the value of $(a+b\,x^2)^2$, and multiply each term of the result by $g\,x^3\,d\,x$; we should in like manner develope the value of $(c+e\,x^2+f\,x^3)^4$, and multiply each term of the result by $a^2\,x^7\,d\,x$; we should then have only to integrate a series of simple quantities, by the fundamental rule.

87. We must however except the case, in which, some one of the exponents being negative, it should happen, after the development and multiplication, that the exponent of the variable in some of the terms became $=-1$; but we should then integrate by logarithms. For example, if we had

$$\frac{a\,d\,x}{x^3}\,(a+b\,x^2)^2, \text{ or } a\,x^{-3}\,d\,x\,(a+b\,x^2)^2\,;$$

we should change it into

$$a\,x^{-3}\,d\,x\,(a^2+2\,a\,b\,x^2+b^2\,x^4),$$

which becomes

$$a^3\,x^{-3}\,d\,x+2\,a^2\,b\,x^{-1}\,d\,x+a\,b^2\,x\,d\,x,$$

of which the two terms $a^3\,x^{-3}\,d\,x$ and $a\,b^2\,x\,d\,x$ have, for their integral, $\frac{-a^3\,x^{-2}}{2}+\frac{a\,b^2\,x^2}{2}$; but the term $2\,a^2\,b\,x^{-1}\,d\,x$, which is the same as $2\,a^2\,b\frac{d\,x}{x}$, is (27) the logarithmic differential of $2\,a^2\,b\,l\,x$; so that

$$\int a\,x^{-3}\,d\,x\,(a+b\,x^2)^2=-\frac{a^3\,x^{-2}}{2}+2\,a^2\,b\log.x+\frac{a\,b^2\,x^2}{2}.$$

88. 3d. If the differential quantity proposed even contains a complex quantity, raised to any power (whether its exponent be positive or negative, whole or fractional), we may still integrate,

if the whole of the terms multiplied by this complex quantity, taken together, be the differential of the complex quantity considered without its total exponent; or if it be this differential multiplied or divided by a constant number. We have only, in that case, to consider the complex quantity in question as a single variable, and apply, word for word, the fundamental rule. For example, $g\,d\,x\,(a+b\,x)^p$ falls under this case, because $g\,d\,x$ is the differential of $a+b\,x$, multiplied by $\frac{g}{b}$, which is a constant quantity; so that, to integrate it, we write

$$\int g\,d\,x\,(a+b\,x)^p = \frac{g\,d\,x\,(a+b\,x)^{p+1}}{(p+1)\,d(a+b\,x)} + C$$

$$= \frac{g\,d\,x\,(a+b\,x)^{p+1}}{(p+1)\,b\,d\,x} + C = \frac{g\,(a+b\,x)^{p+1}}{(p+1)\,b} + C.$$

For, if we differentiate this quantity, we find again

$$g\,d\,x\,(a+b\,x)^p.$$

In like manner, if we examine the differential

$$\frac{a^2\,d\,x+2\,a\,x\,d\,x}{\sqrt{a\,x+x^2}} = (a^2\,d\,x+2\,a\,x\,d\,x)\,(a\,x+x^2)^{-\frac{1}{2}},$$

we shall find that it is integrable, because $a^2\,d\,x+2\,a\,x\,d\,x$ is the differential of $a\,x+x^2$, multiplied by a constant quantity a. By applying the rule, therefore, we have

$$\int (a^2\,d\,x+2\,a\,x\,d\,x)\,(a\,x+x^2)^{-\frac{1}{2}}$$

$$= \frac{(a^2\,d\,x+2\,a\,x\,d\,x)\,(a\,x+x^2)^{\frac{1}{2}}}{\frac{1}{2}\,(a\,d\,x+2\,x\,d\,x)} + C = 2\,a\,(a\,x+x^2)^{\frac{1}{2}} + C.$$

The only case to be excepted is that, in which the exponent of the complex quantity should be -1; when we should integrate by logarithms, as will be seen hereafter.

Of Binomial Differentials which may be integrated algebraically.

89. We understand by a *binomial differential*, one in which the complex quantity, however complex, is some power of a binomial. Thus $g\,x^5\,d\,x\,(a+b\,x^2)^{\frac{3}{5}}$ is a binomial differential. The same may be said of $g\,x^m\,d\,x\,(a+b\,x^n)^p$, which may represent any binomial differential, because, by g, a, b, m, n, p, we understand any imaginable numbers, whether positive or negative.

There are no means of integrating generally every binomial differential. But it is apparent, from what has already been said, that we can integrate a binomial differential $g\, x^m\, d\, x\, (a + b\, x^n)^p$, in the two following cases.

1st. *When* p *is any positive whole number, whatever may be the exponents* m *and* n (85), *with the exception of the case mentioned in* art. 87.

2d. *When* m, *the exponent of* x *out of the binomial, is less by unity than* n, *the exponent of* x *in the binomial*; that is, we may integrate generally, $g\, x^{n-1}\, d\, x\, (a + b\, x^n)^p$, whatever value n and p may have, except the case in which $p = -1$. In fact, $g\, x^{n-1} d\, x$ is the differential of $a + b\, x^n$, multiplied by $\frac{g}{n\, b}$, that is, by a constant quantity; we fall then upon the case mentioned in *art.* 88; and may consequently integrate by the general rule, considering $a + b\, x^n$ as a single quantity.

Besides these cases there are two others, which may be comprehended in one, and which include the preceding; they are the following.

90. 1st. *We may integrate any binomial differential, in which the exponent of* x *out of the binomial, being increased by unity, may be exactly divided by the exponent of* x *in the binomial, and give for a quotient a positive whole number.* The process to be followed in this case, to integrate, and also to show that the principle is general, consists in making the binomial quantity, without its total exponent, equal to a single variable, and expressing the proposed differential by means of this single variable and constants; which may always be done by proceeding as in the following examples.

Let it first be proposed to integrate $g\, x^3\, d\, x\, (a + b\, x^2)^{\frac{4}{5}}$. We see that this differential may be integrated, because the exponent of x out of the binomial, viz. 3, being increased by unity, gives 4, which, divided by 2, the exponent of x in the binomial, gives for a quotient the positive whole number 2.

We make, therefore, $a + b\, x^2 = z$. From this equation we deduce the value of x^2, which is $x^2 = \frac{z - a}{b}$. We observe that $x^3\, d\, x$, which precedes the binomial quantity, results, excepting its constant multiplier, from the differentiation of x^4, the square

of x^2; we therefore square the equation $x^2 = \frac{z-a}{b}$, and find $x^4 = \left(\frac{z-a}{b}\right)^2$; differentiating, we have

$$4\,x^3\,d\,x = 2\left(\frac{z-a}{b}\right)\frac{d\,z}{b},$$

and consequently

$$x^3\,d\,x = \left(\frac{z-a}{b}\right)\frac{d\,z}{2\,b} = \frac{(z-a)\,d\,z}{2\,b^2}.$$

Substituting for $x^3\,d\,x$ and $(a+b\,x^2)$, their values in terms of z, in the expression $g\,x^3\,d\,x\,(a+b\,x^2)^{\frac{4}{5}}$, we find

$$\frac{g\,.\,(z-a)\,d\,z}{2\,b^2}\times z^{\frac{4}{5}} = \frac{g\,z^{\frac{4}{5}+1}\,d\,z}{2\,b^2} - \frac{g\,a\,z^{\frac{4}{5}}\,d\,z}{2\,b^2}.$$

Then

$$\int g\,x^3\,d\,x\,(a+b\,x^2)^{\frac{4}{5}} = \int \frac{g\,z^{\frac{4}{5}+1}\,d\,z}{2\,b^2} - \int \frac{g\,a\,z^{\frac{4}{5}}\,d\,z}{2\,b^2}$$

$$= \frac{g\,z^{\frac{4}{5}+2}}{(\frac{4}{5}+2)\,2\,b^2} - \frac{g\,a\,z^{\frac{4}{5}+1}}{(\frac{4}{5}+1)\,2\,b^2} + C$$

$$= \frac{g\,z^{\frac{4}{5}+1}}{2\,b^2}\left(\frac{z}{\frac{4}{5}+2} - \frac{a}{\frac{4}{5}+1}\right) + C$$

$$= \frac{g\,z^{\frac{4}{5}+1}}{2\,b^2}\left(\frac{5}{14}\,z - \frac{5}{9}\,a\right) + C;$$

substituting then for z, its value $a+b\,x^2$, we have

$$\int g\,x^3\,dx\,(a+bx^2)^{\frac{4}{5}} = \frac{g}{2\,b^2}\,(a+bx^2)^{\frac{4}{5}+1}\left\{\frac{5}{14}(a+b\,x^2) - \frac{5}{9}a\right\} + C.$$

91. We proceed in a similar manner in every other case subject to the same conditions. Let us take, for example,

$$g\,x^8\,d\,x\,(a+b\,x^3)^{-\frac{2}{3}},$$

which must be integrable, since the exponent 8 increased by 1, that is, 9, being divided by 3, the exponent of x in the binomial, gives a positive whole number. We therefore make $a+b\,x^3 = z$; and find $x^3 = \frac{z-a}{b}$, and as $x^8\,d\,x$, which precedes the binomial, is, excepting its constant factor, the differential of x^9, in order to have x^9 we cube the equation $x^3 = \frac{z-a}{b}$; and have $x^9 = \left(\frac{z-a}{b}\right)^3$; then differentiating in order to obtain $x^8\,d\,x$, we have

$$9\,x^8\,d\,x = 3\,.\left(\frac{z-a}{b}\right)^2 . \frac{d\,z}{b},$$

and, consequently,

$$x^8\,d\,x = \left(\frac{z-a}{b}\right)^2 . \frac{d\,z}{3\,b}.$$

The differential $g\,x^8\,d\,x\,(a+b\,x^3)^{-\frac{2}{3}}$ will be therefore changed into $g\,.\left(\frac{z-a}{b}\right)^2 . \frac{d\,z}{3\,b} . z^{-\frac{2}{3}} =$, by expanding

$$\left(\frac{z-a}{b}\right)^2, \text{ to } \frac{g\,z^{2-\frac{2}{3}}\,d\,z}{3\,b^3} - \frac{2\,g\,a\,z^{1-\frac{2}{3}}\,d\,z}{3\,b^3} + \frac{g\,a^2\,z^{-\frac{2}{3}}\,d\,z}{3\,b^3},$$

of which the integral is

$$\frac{g\,z^{3-\frac{2}{3}}}{3\,b^3\,(3-\frac{2}{3})} - \frac{2\,g\,a\,z^{2-\frac{2}{3}}}{3\,b^3(2-\frac{2}{3})} + \frac{g\,a^2\,z^{1-\frac{2}{3}}}{3\,b^3\,(1-\frac{2}{3})} + C,$$

which, separating the common multiplier $\frac{g}{3\,b^3}\,z^{1-\frac{2}{3}}$, is reduced to

$$\frac{g}{3\,b^3}\,z^{1-\frac{2}{3}}\left(\frac{z^2}{3-\frac{2}{3}} - \frac{2\,a\,z}{2-\frac{2}{3}} + \frac{a^2}{1-\frac{2}{3}}\right) + C$$

$$= \frac{g}{3\,b^3}\,z^{1-\frac{2}{3}}\left(\frac{3\,z^2}{7} - \frac{6\,a\,z}{4} + 3\,a^2\right) + C,$$

or finally, substituting for z its value $a+b\,x^3$, we find

$$\int g\,x^8\,d\,x\,(a+b\,x^3)^{-\frac{2}{3}}$$

$$= \frac{g}{3\,b^3}(a+b\,x^3)^{1-\frac{2}{3}}\left\{\frac{3}{7}(a+b\,x^3)^2 - \frac{6}{4}(a+b\,x^3)\,a + 3\,a^2\right\} + C.$$

Such is the method to be pursued whenever the exponent of x out of the binomial, being increased by unity, and divided by the exponent of x in the binomial, will give as quotient a positive whole number.

92. 2d. Although a binomial differential quantity may not fall under the case of which we have just been speaking, it often happens that it may be reduced to it by means of a very simple artifice, which consists in rendering negative the exponent of x in the binomial, when it is positive, and rendering it positive when it is negative. In order to this, we must divide the two terms of the binomial by the power of x in the binomial, and multiply the quantity out of the binomial by this same power of x raised to the power indicated by the total exponent of the binomial. For example, in order to render negative the exponent 2 of x, in the binomial $\qquad g\,x^4\,d\,x\,(a+b\,x^2)^5,$

we divide $a+bx^2$ by x^2, which gives $g\,x^4\,dx\left(\frac{a}{x^2}+b\right)^5$, or $g\,x^4\,dx\,(ax^{-2}+b)^5$; but as the quantity x^2, by which we have divided, is considered as raised to the fifth power, since it is under the total exponent 5 of the binomial, we must, in compensation, multiply the quantity out of the binomial by $(x^2)^5$, that is, by x^{10}, which gives $g\,x^{14}\,dx\,(a\,x^{-2}+b)^5$.

By this preparation, many binomial differentials which would not otherwise be comprehended in the preceding case, will be reduced to it. For example, if it were required to integrate

$$\frac{a^2\,dx}{(a^2+x^2)^{\frac{3}{2}}}=a^2\,dx\,(a^2+x^2)^{-\frac{3}{2}};$$

we perceive that the exponent of x out of the binomial, that is to say, 0, being increased by 1, which makes it 1, cannot be exactly divided by 2, the exponent of x in the binomial; but we must not thence conclude that the proposed quantity is not integrable; for, if we render negative the power of x in the binomial, by writing $a^2\,(x^2)^{-\frac{3}{2}}\,dx\,(a^2\,x^{-2}+1)^{-\frac{3}{2}}$, which is reduced to

$$a^2\,x^{-3}\,dx\,(a^2\,x^{-2}+1)^{-\frac{3}{2}},$$

we see that -3 increased by 1, that is, $-3+1$ or -2, being divided by -2, the exponent of x in the binomial, gives as a quotient a positive whole number; we suppose, therefore,

$$a^2\,x^{-2}+1=z,\text{ whence } x^{-2}=\frac{z-1}{a^2};$$

and as $x^{-3}\,dx$ is, excepting that it wants a constant multiplier, the differential of x^{-2}, we differentiate, which gives

$$-2x^{-3}\,dx=\frac{dz}{a^2},$$

whence we deduce $x^{-3}\,dx=-\frac{dz}{2\,a^2}$. The differential

$$a^2\,x^{-3}\,dx\,(a^2\,x^{-2}+1)^{-\frac{3}{2}},$$

is therefore changed into

$$\frac{-a^2\,dz}{2\,a^2}\cdot z^{-\frac{3}{2}}\text{ or }\frac{-z^{-\frac{3}{2}}\,dz}{2},$$

of which the integral is

$$\frac{-z^{1-\frac{3}{2}}}{2(1-\frac{3}{2})}+C,\text{ or } z^{-\frac{1}{2}}+C,$$

or, substituting for z its value

$$\int \frac{a^2\,dx}{(a^2+x^2)^{\frac{3}{2}}} = (a^2\,x^{-2}+1)^{-\frac{1}{2}} + C$$

$$= \frac{1}{\sqrt{a^2\,x^{-2}+1}} + C = \frac{x}{\sqrt{a^2+x^2}} + C.$$

Thus the process for integrating is the same in this case as in the preceding.

93. We have supposed hitherto that the power of x was found in only one of the terms of the binomial. If it should occur in each, we should reduce the quantity to a form in which it would appear in only one of the terms, by dividing the binomial by one of the two powers of x occurring in its terms, and multiplying the quantity without the binomial by the same power raised to the power indicated by the exponent of the binomial; and that for the reason just given (92), for rendering the exponent negative. Thus, if it were proposed to integrate

$$\frac{a^2\,dx}{x\sqrt{a\,x+x^2}} = a^2\,x^{-1} d\,x\,(a\,x+x^2)^{-\frac{1}{2}},$$

we should change it into $a^2\,x^{-1}\,(x)^{-\frac{1}{2}}\,d\,x\,(a+x)^{-\frac{1}{2}}$, by dividing the binomial by x, and multiplying the quantity without by x raised to the power $-\frac{1}{2}$, which is that of the binomial. This quantity is reduced to $a^2\,x^{-\frac{3}{2}}\,d\,x\,(a+x)^{-\frac{1}{2}}$. If we should apply to it the rule of the first case (90), we should not find it integrable, but by rendering negative the exponent of x in the binomial, we have

$a^2\,x^{-\frac{3}{2}}\,(x)^{-\frac{1}{2}}\,d\,x\,(a\,x^{-1}+1)^{-\frac{1}{2}}$, or $a^2\,x^{-2}\,d\,x\,(a\,x^{-1}+1)^{-\frac{1}{2}}$, which (92) is integrable. Making, therefore, $a\,x^{-1}+1=z$, we have $x^{-1}=\frac{z-1}{a}$; differentiating, we have

$$-x^{-2}\,d\,x = \frac{d\,z}{a}, \text{ or } x^{-2}\,d\,x = \frac{-d\,z}{a};$$

therefore

$$a^2\,x^{-2}\,d\,x\,(a\,x^{-1}+1)^{-\frac{1}{2}} = -a\,d\,z\,.\,z^{-\frac{1}{2}} = -a\,z^{-\frac{1}{2}}\,d\,z,$$

of which the integral is $\frac{-a\,z^{\frac{1}{2}}}{\frac{1}{2}} + C$, or $-2\,a\,z^{\frac{1}{2}} + C$, or, restoring the value of z,

$$\int\left(\frac{a^2\,dx}{x\sqrt{ax+x^2}}\right)=-2a\,(ax^{-1}+1)^{\frac{1}{2}}+C$$

$$=-2a\sqrt{\frac{a}{x}+1}+C.$$

If, upon the examination of a binomial differential, it is found to be comprehended in neither of the two cases above mentioned, it is useless to expect an integral purely algebraical.

As to polynomial differentials, that is, those in which the complex quantity contains three or more terms, they are integrable in the cases mentioned in *art.* 85, *&c.* There are also some other cases in which they admit an algebraic integral, but they are very few and rarely occur. We shall not, therefore, occupy ourselves with them at present.

We shall hereafter show the method of discovering those which are integrable, and those whose integral may be referred to a given integral.

Application of the preceding rules to the quadrature of curves.

94. In order to find the area or quadrature of curved lines, we consider these lines as polygons of an infinite number of sides; and from the extremities M and m of each side (*fig.* 34), we imagine the perpendiculars MP, $m\,p$, let fall upon the axis of the abscissas, dividing the surface into an infinite number of infinitely small trapezia. Then we consider each trapezium, as $PpmM$, as the differential of the finite space APM; because in fact $P\,p\,m\,M = A\,p\,m - APM = d\,(APM)$ (6). It is therefore only required to express algebraically the little trapezium $PpmM$, and then integrate this expression by means of the preceding rules.

But, in considering $P\,p\,m\,M$ as the differential of the surface, it must be observed, that it is no more the differential of the surface reckoned from A, the origin of the abscissas, than it is the differential of any other surface $KPML$ reckoned from a fixed and determinate point K; since we have equally

$$P\,p\,m\,M = K\,p\,m\,L - KPML = d\,(KPML).$$

When we integrate, therefore, we have no right to refer the integral given directly by the calculus to the space APM, rather than to any other space $KPLM$, which differs from it by a determinate and constant space KAL. We must therefore add to the integral

found by the calculus, a constant quantity expressing that by which the space proposed to be determined differs from that given directly by the calculus. It will be seen in the following examples, how this constant quantity is determined. Let us, in the first place, find the expression for the space $P\,p\,m\,M$.

Calling AP, x; PM, y; we have $P\,p = d\,x$, $p\,m = y + d\,y$. The surface of the trapezium $P\,p\,m\,M$ is (*Geo.* 178)

$$\frac{PM + p\,m}{2} \times P\,p = \frac{2\,y + d\,y}{2} \times d\,x = y\,d\,x + \frac{d\,y\,d\,x}{2}.$$

But to indicate that $d\,y$ and $d\,x$ are infinitely small, we must reject $\frac{d\,y\,d\,x}{2}$, which is infinitely small, compared with $y\,d\,x$; we have therefore $y\,d\,x$ *as the general expression of the differential of the element of the surface of any curve.*

In order to apply this formula to a proposed surface, whose equation is given, we must deduce from this equation the value of y, which we substitute in the formula $y\,d\,x$, we then have a quantity in terms of x and $d\,x$, which, when it can be integrated by the preceding rules, will give, with the addition of a constant quantity, the expression of the surface of this curve, reckoned from any point we please. We have then only to determine the constant quantity, which is done by expressing from what point we choose to estimate the surface. We shall now illustrate this theory by examples.

Let us take, for the first example, the common parabola, which has for its equation $y^2 = p\,x$. We have $y = \sqrt{p\,x} = p^{\frac{1}{2}}\,x^{\frac{1}{2}}$; therefore

$$y\,d\,x = p^{\frac{1}{2}}\,x^{\frac{1}{2}}\,d\,x.$$

But (83) $\int p^{\frac{1}{2}}\,x^{\frac{1}{2}}\,d\,x = \frac{p^{\frac{1}{2}}\,x^{\frac{3}{2}}\,d\,x}{\frac{3}{2}d\,x} + C = \frac{2}{3}\,p^{\frac{1}{2}}\,x^{\frac{3}{2}} + C$;

this last expression then is that of the surface of the parabola; so that, knowing the abscissa x, and the parameter p, we shall have the value of the space APM; or, of the space $KPLM$ estimated from a determinate point K, if the constant C be determinate, that is, if this integral express actually from what point we estimate.

Suppose, then, that we wish to estimate the space from the point A; in that case we have

$$APM = \frac{2}{3}p^{\frac{1}{2}}\,x^{\frac{3}{2}} + C.$$

In order to know what becomes of the constant on this hypothesis, we observe, that when x becomes 0, the space APM is also zero; in that case, the equation is reduced to $0 = 0 + C$; wherefore $C = 0$; in order then that the integral may express the space estimated from the point A, the constant C must be zero; that is, we have no constant to add, and we have, generally, the indefinite space $APM = \frac{2}{3}p^{\frac{1}{2}} x^{\frac{3}{2}}$.

But if we wished to estimate the space from the point K, such that $AK = b$, (b being a known quantity); we should have

$$KPLM = \tfrac{2}{3}p^{\frac{1}{2}} x^{\frac{3}{2}} + C;$$

now this space $KPLM$ becomes zero, when AP or $x = b$; we have therefore, in that case, $0 = \frac{2}{3} p^{\frac{1}{2}} b^{\frac{3}{2}} + C$; therefore

$$C = -\tfrac{2}{3} p^{\frac{1}{2}} b^{\frac{3}{2}},$$

and consequently

$$KPLM = \tfrac{2}{3} p^{\frac{1}{2}} x^{\frac{3}{2}} - \tfrac{2}{3} p^{\frac{1}{2}} b^{\frac{3}{2}}.$$

We thus see what purpose is served by the constant, which is added in integrating, and that the conditions of the equation alone can determine it.

We observe that $\frac{2}{3} p^{\frac{1}{2}} x^{\frac{3}{2}} = \frac{2}{3} p^{\frac{1}{2}} x^{\frac{1}{2}} \times x$; but $p^{\frac{1}{2}} x^{\frac{1}{2}} = y$; therefore $\frac{2}{3} p^{\frac{1}{2}} x^{\frac{3}{2}}$, or $\frac{2}{3} p^{\frac{1}{2}} x^{\frac{1}{2}} \times x = \frac{2}{3} y\, x$; since therefore $\frac{2}{3} p^{\frac{1}{2}} x^{\frac{3}{2}}$ expresses the space APM, this space will also have for its expression $\frac{2}{3}\, x\, y$, that is, $\frac{2}{3}\, AP \times PM$, or $\frac{2}{3}$ of the rectangle $APMO$, whatever AP may be.

In like manner

$$\tfrac{2}{3} p^{\frac{1}{2}} b^{\frac{3}{2}} = \tfrac{2}{3} p^{\frac{1}{2}} b^{\frac{1}{2}} \times b;$$

but, when $x = AK = b$, the equation $y^2 = p\, x$ gives $y^2 = p\, b$, and consequently $y = p^{\frac{1}{2}} b^{\frac{1}{2}}$; that is, $KL = p^{\frac{1}{2}} b^{\frac{1}{2}}$; therefore $\frac{2}{3}\, p^{\frac{1}{2}} b^{\frac{3}{2}}$ or $\frac{2}{3}\, p^{\frac{1}{2}} b^{\frac{1}{2}} \times b = \frac{2}{3}\, KL \times AK$; therefore since the space $KPLM$ is represented by $\frac{2}{3}\, p^{\frac{1}{2}}\, x^{\frac{3}{2}} - \frac{2}{3}\, p^{\frac{1}{2}}\, b^{\frac{3}{2}}$, it will have also for its expression $\frac{2}{3}\, AP \times PM - \frac{2}{3}\, AK \times KL$, that is, $\frac{2}{3}\, APMO - \frac{2}{3}\, AKLI$.

The parabola is the only one of the four conic sections susceptible of being squared.

Let us take, as a second example, parabolas of all kinds, whose general equation (29) is $y^{m+n} = a^m x^n$;
we have therefore

$$y = \sqrt[m+n]{a^m x^n} = a^{\frac{m}{m+n}} x^{\frac{n}{m+n}}$$

then

$$y\,dx = a^{\frac{m}{m+n}} x^{\frac{n}{m+n}}\,dx;$$

and

$$\int y\,dx = \frac{a^{\frac{m}{m+n}} x^{\frac{n}{m+n}+1}}{\frac{n}{m+n}+1} + C = \frac{m+n}{m+2n} a^{\frac{m}{m+n}} x^{\frac{n}{m+n}+1} + C$$

$$= \frac{m+n}{m+2n} x\, a^{\frac{m}{m+n}} x^{\frac{n}{m+n}} + C = \frac{m+n}{m+2n} x\,y + C.$$

So that if we wish to estimate the space APM from A, the origin of the abscisses x (*fig.* 35), which requires the integral to be zero when APM is zero, and when consequently x is zero, then the constant C is zero, and we have simply

$$APM = \frac{m+n}{m+2n} x\,y;$$

that is, the space APM is always a determinate portion of the product $x\,y$ or of the rectangle $APMO$, it is that portion of it expressed by the fraction $\frac{m+n}{m+2n}$, the value of which depends on the values of m and n: that is, on the degree of the parabola. Thus all parabolas are susceptible of being squared.

It will also be found that all the hyperbolas referred to their asymptotes, (except the common hyperbola,) are susceptible of being squared. But as, in the determination of the constant, we sometimes find an indefinite quantity, it may not be useless to examine here, in what sense it is to be understood. Let, therefore, $y^m = a^{m+n} x^{-n}$, be the equation of these curves; we have

$$y = a^{\frac{m+n}{m}} x^{\frac{-n}{m}};$$

therefore

$$y\,dx = a^{\frac{m+n}{m}} x^{\frac{-n}{m}}\,dx,$$

and

$$\int y\,dx = \frac{a^{\frac{m+n}{m}} x^{1-\frac{n}{m}}}{1-\frac{n}{m}} + C = \frac{m}{m-n} a^{\frac{m+n}{m}} x^{1-\frac{n}{m}} + C;$$

a quantity in which there is no difficulty in determining the constant when m exceeds n. But when m is less than n, we find an infinite quantity for the constant, if we estimate the space from the origin of x; and a finite quantity, if we estimate from any other point. Let us suppose, for example, $m = 1$, and $n = 2$; in which case the equation is $y = a^3 x^{-2}$; the surface is then reduced to $-a^3 x^{-1} + C$, or $C - \frac{a^3}{x}$. If then we wish to estimate the space from A, the origin of the abscissas x, $C - \frac{a^3}{x}$ must be zero, when $x = 0$; that is, $C - \frac{a^3}{0} = 0$, and consequently $C = \frac{a^3}{0}$; that is, is infinite. If, on the contrary, we wish to estimate the space from the point K, such that $KA = b$, we have $C - \frac{a^3}{b} = 0$, which gives $C = \frac{a^3}{b}$, and consequently this space $= \frac{a^3}{b} - \frac{a^3}{x}$. This result is to be understood in the following manner.

The curve which has for its equation $y = a^3 x^{-2}$ or $y = \frac{a^3}{x^2}$ extends to infinity along the asymptotes AZ, AY (*fig.* 36); but approaches much more nearly to the asymptote AZ than to the asymptote AY; for, when x is infinite, y is infinitely small of the second order, but when y is infinite, x is only infinitely small of the order $\frac{1}{2}$; if therefore we estimate the spaces from the asymptote AY, they are infinite, because the space comprehended between this asymptote and the infinite branch BS is infinite. On the contrary, the spaces comprehended between the branch BM and the asymptote AZ to infinity, have a finite value, because, after a very short interval, the branch approaches its asymptote very rapidly, so that the infinitely long space $KLMOZ$ has for its expression $\frac{a^3}{b}$, and $PMOZ = \frac{a^3}{x}$; and consequently $KLMP = \frac{a^3}{b} - \frac{a^3}{x}$. Whence it follows, that though we may not obtain the spaces estimated from AY, we may, nevertheless, find the spaces $KLMP$, estimated from a point K, taken as near as we please to AY.

4 Let us take, as a third example, the curve which has for its equation $y = \frac{a^2 x - x^3}{a^2}$, and which will be found to have the figure delineated (*fig.* 37), by giving successively to x arbitrary values, and a determinate value to a.

We shall have then

$$y\,dx = \frac{a^2 x\,dx - x^3\,dx}{a^2},$$

and (83),

$$\int y\, d\, x = APM = \frac{2\, a^2\, x^2 - x^4}{4\, a^2} + C;$$

and if we wish to estimate the space APM from the point A, the origin of the abscissas x, this integral must become 0 with x, which shows that the constant $C =$ zero. So that the indefinite space APM is simply $\frac{2\, a^2\, x^2 - x^4}{4\, a^2}$. And, in general, if the value of y is composed, as in the present case, of simple terms only, it will always be easy to find the surface.

95. We have seen† how, by the assistance of algebra, we may imitate any perimeter whatever $ABCD$ (*fig.* 38), by causing to pass through a certain number of its points A, B, C, D, a curve line whose equation shall have the form

$$y = a + b\, x + c\, x^2 + e\, x^3 + f\, x^4, \text{ \&c.},$$

and how, to this end, we may determine a, b, c, &c. Let us now suppose that it were required to find the surface $ABCDLK$, although we had not the equation of the curve $ABCD$.

We should draw through a certain number of points, A, B, C, D, a curve $A\, e\, B\, f\, C\, g\, D$, which would coincide with the former the more perfectly according as we should take a greater number of points. And, as we should then have the equation of this latter curve, we might consider it as the equation of the curve $ABCD$, at least for the extent $ABCD$. But this equation being then of the form $y = a + b\, x + c\, x^2 + ex^3$, &c.
of which all the terms are simple, we should easily find the surface according to what has just been said. We may apply this method to the measure of the surfaces of the section of vessels.

In general, it may be applied to find, by approximation, the surfaces of curves and the approximate integral of quantities which cannot be integrated exactly. Indeed, every differential may be regarded as expressing the element of the surface of a curve, whose ordinate is equal to the total factor of $d\, x$. For example, $d\, x \sqrt{a^2 + x^2}$ is the element of the surface of the curve, which has for its ordinate $y = \sqrt{a^2 + x^2}$. Thus, calculating, by means of this equation, some values of y for certain values of x, and drawing through the extremities of these ordinates a curve of the nature of those just mentioned, the surface of these latter being found, would be the approximate value of the integral of $d\, x \sqrt{a^2 + x^2}$ for the extent of x.

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† See note at the end.

Let us take another example. It shall be that of the surface of the curve which has for its equation $a^5 y^2 = x^4 (a^3 - x^3)$; this equation gives

$$y = \pm \sqrt{\frac{x^4 (a^3 - x^3)}{a^5}} = \pm \frac{x^2}{a^2 \sqrt{a}} \sqrt{a^3 - x^3};$$

therefore (taking only one of the values of y),

$$y\,d\,x = \frac{x^2\,d\,x}{a^2 \sqrt{a}} \sqrt{a^3 - x^3} = \frac{x^2\,d\,x}{a^2 \sqrt{a}} (a^3 - x^3)^{\frac{1}{2}};$$

now this quantity is integrable (89) because $x^2\,d\,x$ is the differential of the term x^3 in the binomial, divided by the constant number 3.

Therefore (89) we have

$$\int y\,d\,x = \frac{x^2\,d\,x\,(a^3 - x^3)^{\frac{3}{2}}}{\frac{3}{2}\,a^2 \sqrt{a}. - 3\,x^2\,d\,x} + C = -\frac{2\,(a^3 - x^3)^{\frac{3}{2}}}{9\,a^2 \sqrt{a}} + C.$$

With regard to the constant C, it will be determined by deciding from what point we wish to reckon the surface.

We may also find the surface of curves by decomposing them into triangles instead of trapeziums. For example, we might find the surface of the segment ANQ (*fig.* 34), by considering it as composed of an infinite number of infinitely small triangles, such as AQq. This triangle would have for its expression $\frac{A\,q \times Q\,t}{2}$, by letting fall the perpendicular $Q\,t$, or, which amounts to the same thing, by describing from the centre A, with the radius AQ, the infinitely small arc $Q\,t$. Then calling AQ, t, and the arc $Q\,t$, $d\,x$, we should have $A\,q = t + d\,t$, and consequently the triangle

$$AQq = \frac{t + d\,t}{2}\,d.x = \frac{t\,d\,x}{2} + \frac{d\,t\,d\,x}{2}, \text{ that is, } = \frac{t\,d\,x}{2},$$

rejecting the term $\frac{d\,t\,d\,x}{2}$, in order to indicate that $d\,x$ and $d\,t$ are infinitely small. Nothing more would be required than to have the equation in terms of x and t, in order to substitute in place of t its value in x, and integrate.

Application to the rectification of Curved Lines.

5

96. To rectify a curved line is to determine its length or to assign a straight line which shall be equal to it, or to any proposed arc of it. The following is the method to be used, when the rectification is possible.

Considering the curve AM (*fig.* 34) as a polygon of an infi-

nite number of sides, the small side $M\,m$ may be regarded as the differential of the arc AM, because

$$M\,m = A\,m - AM = d\,(AM).$$

Now, drawing $M\,r$ parallel to AP, we have

$$M\,m = \sqrt{M\,r^2 + r\,m^2} = \sqrt{d\,x^2 + d\,y^2}\,;$$

it is therefore only necessary to integrate $\sqrt{d\,x^2 + d\,y^2}$. In order to this, we differentiate the equation of the curve, and having deduced from it the value of $d\,y$, expressed in terms of x and $d\,x$, or that of $d\,x$ in terms of y and $d\,y$, we substitute it in the expression $\sqrt{d\,x^2 + d\,y^2}$, which will thus contain only terms of x and $d\,x$ or of y and $d\,y$; we then remove $d\,x^2$ or $d\,y^2$ from under the radical sign (*Alg.* 130) and integrate.

To give an example, let us take, among the parabolas expressed generally by $y^{m+n} = a^m\,x^n$, that which has for its particular equation $y^3 = a\,x^2$; we deduce, $x^2 = \dfrac{y^3}{a}$; and $x = \dfrac{y^{\frac{3}{2}}}{a^{\frac{1}{2}}}$; whence

$$d\,x = \frac{\frac{3}{2}\,y^{\frac{1}{2}}\,d\,y}{a^{\frac{1}{2}}}, \text{ and } d\,x^2 = \frac{9}{4}\,\frac{y\,d\,y^2}{a}\,;$$

therefore

$$\sqrt{d\,x^2 + d\,y^2} = \sqrt{d\,y^2 + \frac{9\,y\,d\,y^2}{4\,a}} = d\,y\,\sqrt{1 + \frac{9\,y}{4\,a}}.$$

Now this quantity is easily integrated (90), since the exponent of y out of the binomial is less by unity than its exponent in the binomial. Therefore

$$\int d\,y\,\sqrt{1 + \frac{9\,y}{4\,a}} = \int d\,y\left(1 + \frac{9\,y}{4\,a}\right)^{\frac{1}{2}}$$

$$= \frac{d\,y\left(1 + \dfrac{9\,y}{4\,a}\right)^{\frac{3}{2}}}{\dfrac{3}{2}\cdot\dfrac{9\,d\,y}{4\,a}} + C = \frac{8\,a}{27}\cdot\left(1 + \frac{9\,y}{4\,a}\right)^{\frac{3}{2}} + C.$$

The constant C is determined as follows. If we wish to estimate the arcs AM from the point A, the origin of y, the integral or value of the arc AM must become zero at the same time that $y = 0$. But when $y = 0$, the integral is reduced to

$$\frac{8\,a}{27} \times (1)^{\frac{3}{2}} + C = \frac{8\,a}{27} + C;$$

therefore $\dfrac{8\,a}{27} + C = 0$; whence $C = -\dfrac{8\,a}{27}$. Therefore the

length of any arc AM, reckoned from the vertex A, is

$$\frac{8a}{27}\left(1+\frac{9y}{4a}\right)^{\frac{3}{2}}-\frac{8a}{27}.$$

If it be required to find what other parabolas are susceptible of being rectified, we may proceed in the following manner. The equation $y^{m+n}=a^m x^n$, which belongs to these curves, gives

$$y=a^{\frac{m}{m+n}}x^{\frac{n}{m+n}}.$$

Let us take $\frac{m}{m+n}=k$, and $\frac{n}{m+n}=p$; we have $y=a^k x^p$; whence

$$dy=p\,a^k x^{p-1}dx,\text{ and }dy^2=p^2a^{2k}x^{2p-2}dx^2;$$

wherefore

$$\sqrt{dx^2+dy^2}=\sqrt{dx^2+p^2a^{2k}x^{2p-2}dx^2}$$
$$=dx\sqrt{1+p^2a^{2k}x^{2p-2}},$$

a quantity which is not integrable in this state except when $2p-2=1$. But, if we change the sign of the exponent of x under the radical, we have $\quad x^{p-1}dx\sqrt{x^{-2p+2}+p^2a^{2k}}$,
which (91) is integrable if $p-1$ increased by unity and divided by $-2p+2$, gives a positive whole number; that is, if $\frac{p}{-2p+2}=t$, t being a positive whole number. Whence we deduce

$$p=-2tp+2t,\text{ or }p=\frac{2t}{2t+1};$$

but $p=\frac{n}{m+n}$, wherefore $\frac{n}{m+n}=\frac{2t}{2t+1}$; whence $m=\frac{n}{2t}$; thus the parabolas which may be rectified, are those comprised in the equation $\quad y^{\frac{2nt+n}{2t}}=a^{\frac{n}{2t}}x^n$;
or, extracting the root of the degree n,

$$y^{\frac{2t+1}{2t}}=a^{\frac{1}{2t}}x,$$

t being any whole number whatever.

Application to Curved Surfaces.

97. We shall confine ourselves to the surfaces of the solids of revolution. We call by this name the solids conceived to be generated by the revolution of a curve AM (*fig.* 39) about a straight line AP.

We imagine that while AM revolves about AP, the small side Mm describes a zone or portion of a truncated cone, which is the element of the surface, and which (*Geo.* 531), is equal to the product of Mm, by the circumference which has for its radius the perpendicular let fall from the middle of Mm upon AP, or which is the same thing, since $M\,m$ is infinitely small, by the circumference which has for its radius PM. Now the arc

$$Mm = \sqrt{dx^2 + dy^2};$$

and if we represent by $r : c$ the ratio of the radius of a circle to its circumference, we have $r : c :: y$ is to the circumference which has PM for its radius, which will be $\frac{cy}{r}$; whence we have

$$\frac{cy}{r}\sqrt{dx^2 + dy^2}$$

for the element of the surface of the solids of revolution.

98. Let us suppose, as an example, that the surface of the sphere is required. The generating circle AMB (*fig.* 40) has for its equation $y^2 = ax - x^2$, calling AP, x, and PM, y. Whence

$$y = \sqrt{ax - x^2}, \text{ and } dy = \frac{\frac{1}{2}a\,dx - x\,dx}{\sqrt{ax - x^2}}$$

wherefore

$$dy^2 = \frac{\frac{1}{4}a^2\,dx^2 - ax\,dx^2 + x^2\,dx^2}{ax - x^2};$$

therefore

$$\sqrt{dx^2 + dy^2} = \sqrt{dx^2 + \frac{\frac{1}{4}a^2\,dx^2 - ax\,dx^2 + x^2\,dx^2}{ax - x^2}}$$

$$= \sqrt{\frac{ax\,dx^2 - x^2\,dx^2 + \frac{1}{4}a^2\,dx^2 - ax\,dx^2 + x^2\,dx^2}{ax - x^2}} = \frac{\frac{1}{2}a\,dx}{\sqrt{ax - x^2}}.$$

Substituting, then, in the formula $\frac{cy}{r}\sqrt{dx^2 + dy^2}$, for y and $\sqrt{dx^2 + dy^2}$, their values just found, we have

$$c\frac{\sqrt{ax - x^2}}{r} \times \frac{\frac{1}{2}a\,dx}{\sqrt{ax - x^2}},$$

which becomes $\frac{\frac{1}{2}acdx}{r}$, and

$$\int \frac{\frac{1}{2}acdx}{r} = \frac{\frac{1}{2}acx}{r} + C.$$

or simply $\frac{\frac{1}{2}acx}{r}$, if we reckon the surface from the point A. Now as $\frac{\frac{1}{2}ac}{r}$ expresses the circumference of the circle whose

radius is $\frac{1}{2}a$, since $r : c :: \frac{1}{2}a : \frac{\frac{1}{2}ac}{r}$; $\frac{\frac{1}{2}ac}{r}x$ expresses the surface of a sphere whose diameter is x; which agrees with what has already been found (*Geo.* 535.)

99. To find the surface of the paraboloid, (which is the solid generated by the revolution of the parabola AM (*fig.* 39) about its axis), we have the equation $y^2 = px$; whence

$$x = \frac{y^2}{p},\ dx = \frac{2y\,dy}{p},\ \text{and}\ dx^2 = \frac{4y^2\,dy^2}{p^2};$$

whence

$$\sqrt{dx^2 + dy^2} = \sqrt{dy^2 + \frac{4y^2\,dy^2}{p^2}} = dy\sqrt{1 + \frac{4y^2}{p^2}};$$

wherefore

$$\frac{cy}{r}\sqrt{dx^2 + dy^2} = \frac{cy\,dy}{r}\sqrt{1 + \frac{4y^2}{p^2}};$$

which, being integrated, (90) gives

$$\frac{\frac{cy\,dy}{r}\left(1 + \frac{4y^2}{p^2}\right)^{\frac{3}{2}}}{\frac{3}{2}.\frac{8y\,dy}{p^2}} + C = \frac{p^2c}{12r}\left(1 + \frac{4y^2}{p^2}\right)^{\frac{3}{2}} + C.$$

Now, in order that this quantity should express the surface reckoned from the vertex A, it must become zero, when $y = 0$; but, in that case, it becomes $\frac{p^2c}{12r}(1)^{\frac{3}{2}} + C$, or $\frac{p^2c}{12r} + C$; whence $\frac{p^2c}{12r} + C = 0$; that is $C = -\frac{p^2c}{12r}$;

wherefore, the surface of the indefinite paraboloid $AMLA$ is

$$\frac{p^2c}{12r}\left(1 + \frac{4y^2}{p^2}\right)^{\frac{3}{2}} - \frac{p^2c}{12r}.$$

Application to the measure of solidity.

100. In order to measure the solidity of bodies, we may imagine them composed of infinitely thin parallel segments, or of an infinite number of pyramids, whose summits unite in the same point. In the first way of viewing the subject, the difference of the two opposite surfaces, which terminate each segment is infinitely small, and must consequently be omitted in the calculus, if we would indicate that the segment is infinitely thin. Thence it follows that we must take, as the expression of the solidity of this segment, the product of one of its opposite bases by its infinitely

small altitude. If, for example, we consider the pyramid $SABC$, (*fig.* 41) as composed of infinitely thin segments, like $a\,b\,c\,d\,e\,f$; we may take, as the measure of this segment, the product of the surface $a\,b\,c$ or $d\,e\,f$, by the thickness of this segment.

In like manner, if we consider the solid generated by the revolution of the curve AM about the straight line AP (*fig.* 39), as composed of infinitely thin parallel segments, we must take as the measure of each segment, the product of the surface of the circle, which has for its radius PM, by the thickness $P\,p$.

This principle being laid down, we thus estimate the solidity of the whole body. We consider each segment as being the differential of the solid, because the segment $M\,m\,l\,L$ is in fact

$$= A\,m\,l\,A - AMLA = d\,(AMLA);$$

and having determined the algebraical expression of this segment, we integrate.

Let it be required, for example, to find the solidity of the pyramid $SABC$ (*fig.* 41). Supposing the surface of the base ABC to be equal to the known quantity b^2, and its altitude $S\,T = h$, we represent by x the distance $S\,t$ of any section; which gives $d\,x$ for the thickness of this segment. The surface $a\,b\,c$ is found by the following proportion (*Geo.* 409);

$$ST^2 : St^2 :: ABC : a\,b\,c;$$

that is,

$$h^2 : x^2 :: b^2 : a\,b\,c = \frac{b^2\,x^2}{h^2};$$

thus the solidity of the segment will be

$$\frac{b^2\,x^2\,d\,x}{h^2} \text{ and } \int \frac{b^2\,x^2\,d\,x}{h^2} = \frac{b^2\,x^3}{3\,h^2} + C = \frac{b^2\,x^3}{3\,h^2},$$

if we reckon the solidity from the vertex S. This quantity, which expresses the solidity of any pyramidal portion $S\,a\,b\,c$, is the same as $\frac{b^2\,x^2}{h^2} \times \frac{x}{3}$, which is $a\,b\,c \times \frac{St}{3}$; which agrees with what has been demonstrated (*Geo.* 416).

101. As to the solids of revolution, we may find a general expression for the elementary segment or differential. For, supposing that $r : c$ expresses the ratio of the radius to the circumference, we shall find the circumference, which has PM (*fig.* 39) or y for its radius, by the following proportion,

$$r : c :: y : \frac{c\,y}{r};$$

if we multiply this value $\frac{cy}{r}$ of the circumference, which has PM for its radius, by $\frac{1}{2}y$ or half of the radius, we have $\frac{cy^2}{2r}$, for the surface, which, being multiplied by the thickness Pp or dx, gives $\frac{cy^2dx}{2r}$ for the expression of the element of the solidity of every solid of revolution. To employ it in any particular case, we have only to substitute instead of y its value in terms of x derived from the equation of the generating curve AM, and integrate.

102. Take, as an example, the spheroid generated by the revolution of the ellipse about its major axis (*fig.* 42). The equation of the ellipse is $y^2 = \frac{b^2}{a^2}(ax - x^2)$ calling AP, x, and PM, y, and the axes AB and Dd, a and b; we have then,

$$\frac{cy^2dx}{2r} = \frac{cb^2}{2ra^2}(ax - x^2)\,dx = \frac{cb^2}{2ra^2}(ax\,dx - x^2\,dx),$$

of which the integral

$$= \frac{cb^2}{2ra^2}\left(\frac{ax^2}{2} - \frac{x^3}{3}\right) + C = \frac{cb^2}{2ra^2}\left(\frac{ax^2}{2} - \frac{x^3}{3}\right),$$

when the solidity is reckoned from the point A.

In order to obtain the entire spheroid, we suppose $x = AB = a$, and we have $\frac{cb^2}{2ra^2} \times \left(\frac{a^3}{2} - \frac{a^3}{3}\right)$, which is reduced to $\frac{cab^2}{12r}$, which is the same thing as $\frac{cb^2}{4r} \times \frac{1}{3}a$, or $\frac{cb^2}{8r} \times \frac{2}{3}a$; now $\frac{cb^2}{8r}$ expresses the surface of the circle, which has b or Dd for its diameter, since (*Geom.* 287,) $r^2 : \frac{cr}{2} :: \frac{b^2}{4} : \frac{cb^2}{8r}$; and $\frac{cb^2}{8r} \times a$ would consequently express the solidity of the cylinder circumscribed about the ellipsoid. Therefore, since the solidity of the ellipsoid is here $\frac{cb^2}{8r} \times \frac{2}{3}a$, we conclude that it is two thirds of that of the circumscribed cylinder. And as the sphere is only an ellipsoid, of which the two axes are equal, the sphere is therefore $\frac{2}{3}$ of the circumscribed cylinder, which agrees with what has been demonstrated (*Geom.* 549).

103. If we wish to determine the solidity, reckoning from a determinate point K such that $AK = e$; we take the general integral $\frac{cb^2}{2ra^2}\left(\frac{ax^2}{2} - \frac{x^3}{3}\right) + C$; and as the solidity is to be reckoned

from K, this integral must become 0 at that point, that is, when $x = e$; in this case it becomes $\frac{c\,b^2}{2\,r\,a^2}\left(\frac{a\,e^2}{2} - \frac{e^3}{3}\right) + C$; whence

$$\frac{c\,b^2}{2\,r\,a^2}\left(\frac{a\,e^2}{2} - \frac{e^3}{3}\right) + C = 0,$$

and consequently

$$C = -\frac{c\,b^2}{2\,r\,a^2}\left(\frac{a\,e^2}{2} - \frac{e^3}{3}\right);$$

thus the solidity, reckoned from the point K, has for its expression

$$\frac{c\,b^2}{2\,r\,a^2}\left(\frac{a\,x^2}{2} - \frac{x^3}{3}\right) - \frac{c\,b^2}{2\,r\,a^2}\left(\frac{a\,e^2}{2} - \frac{e^3}{3}\right).$$

Such, therefore, is the value of the segment of an elliptical spheroid, comprehended between two parallel planes, perpendicular to the axis, and at the distance $x - e$ from each other.

We may, by this formula, calculate the solidity and consequently the weight of the masts and yards of vessels, which are portions of elliptical spheroids. The same formula serves also to measure the capacity of casks, whose external surface may be regarded as a portion of such a spheroid.

104. Let us take, as a second example, the paraboloid (*fig.* 39). The equation of the parabola is $y^2 = px$, thus the formula $\frac{c\,y^2\,d\,x}{2\,r}$ becomes $\frac{c\,p\,x\,d\,x}{2\,r}$, whose integral is

$$\frac{c\,p\,x^2}{4\,r} + C = \frac{c\,p\,x}{2\,r} \times \frac{x}{2} + C = \frac{c\,y^2}{2\,r} \times \frac{x}{2} + C,$$

by substituting for $p\,x$ its value y^2. If we reckon the solid from the point A, as it is zero when $x = 0$, the constant C must be zero, and the solidity is reduced to $\frac{c\,y^2}{2\,r} \times \frac{x}{2}$; now $\frac{c\,y^2}{2\,r}$ expresses the surface of the circle, which has PM for its radius, that is, the base of the paraboloid $AMLA$; therefore the paraboloid is half the product of its base by its altitude; it is therefore half of the cylinder of the same base and altitude.

If we wish to reckon the solidity from the known point K, such that $AK = e$; then, as the solidity must be zero at the point K, that is, when $x = e$, the general integral must be zero at the same time; that is, $\frac{c\,p\,x^2}{4\,r} + C$, becoming $\frac{c\,p\,e^2}{4\,r} + C$, equals zero,

wherefore

$$\frac{c\,p\,e^2}{4\,r} + C = 0,$$

and consequently

$$C = -\frac{c\,p\,e^2}{4\,r};$$

whence, the solidity of the segment of a paraboloid comprehended between two parallel planes, whose distances from the vertex are x and e, is $\frac{c\,p\,x^2}{4\,r} - \frac{c\,p\,e^2}{4\,r}$. This formula may serve to estimate the excavations of mines.

Ema

105. We may take, as another example, the hyperboloid or solid generated by the revolution of the hyperbola about one of its axes. We may also take the ellipsoid generated by the revolution of the ellipse about its lesser or minor axis, which is called the *Flattened Ellipsoid* (Oblate Spheroid). That generated by its revolution about the greater or major axis is called *Elongated Ellipsoid* (Prolate Spheroid). We should, in like manner, find that the flattened ellipsoid is $\frac{2}{3}$ of the circumscribed cylinder: that is, that a and b, being the greater and less axis of the generating ellipse, the elongated spheroid has for its solidity $\frac{c\,a\,b^2}{12\,r}$, and the flattened spheroid has for its solidity $\frac{c\,a^2\,b}{12\,r}$; thus the elongated spheroid is to the flattened spheroid $:: \frac{c\,a\,b^2}{12\,r} : \frac{c\,a^2\,b}{12\,r} :: b : a$, as the lesser axis to the greater.

We shall now leave the solids of revolution.

But in order to accustom beginners to extend the application of these methods, we shall give two additional examples.

106. In the first, it is proposed to find the solidity of a cylindrical ungula formed by cutting a cylinder by a plane oblique to its base, and which, for greater simplicity, we will suppose to pass through the centre. It is the solid $ADBE$ represented (*fig.* 43).

If we conceive this solid to be cut by parallel planes infinitely near each other, and perpendicular to the base AEB (*fig.* 44), the sections will be similar triangles, whose surfaces will consequently be as the squares of their homologous sides. Thus, calling CE the radius of the base, r, the altitude DE, h, and PM, the base of the triangle PMN, y, we have

$$CED : PMN :: r^2 : y^2;$$

now

$$CED = \frac{rh}{2};$$

therefore

$$PMN = \frac{rhy^2}{2rr} = \frac{hy^2}{2r};$$

calling, therefore, AP, x, which gives dx for Pp, the thickness of the segment comprehended between two contiguous planes, we have $\frac{hy^2dx}{2r}$ for this segment. Now y is an ordinate of the circle which serves as base, and we have consequently $y^2 = 2rx - x^2$. The elementary segment becomes then $\frac{hdx(2rx - x^2)}{2r}$, or $\frac{h}{2r}(2rxdx - x^2dx)$, of which the integral, reckoning from the point A, is $\frac{h}{2r}\left(rx^2 - \frac{x^3}{3}\right)$. Therefore, to obtain the whole solidity, we have only to suppose $x = 2r$, which gives

$$\frac{h}{2r}\left(4r^3 - \frac{8r^3}{3}\right) = \tfrac{2}{3}hr^2 = \frac{hr}{2} \times \tfrac{4}{3}r = CED \times \tfrac{4}{3}AC$$
$$= CED \times \tfrac{2}{3}AB,$$

that is, two thirds of the prism, which should have the triangle CED for its base, and the diameter AB for its altitude. This may serve to measure the solidity of fortifications.

107. As a second example, we shall investigate the solidity of a segment of an elongated ellipsöid, comprehended between two planes parallel to each other and to the greater axis.

Before proceeding in this investigation, it must be demonstrated that the sections of an ellipsoid, made parallel to the greater axis, are ellipses similar to the generating ellipse of the solid, that is, that their axes have the same ratio to each other as the axes of that ellipse.

To this end, we conceive the ellipsoid cut by a plane, which for the sake of preciseness, we suppose vertical, and passing through the greater axis AB (*fig.* 45). The section will be the ellipse $ADBE$ equal to the generating ellipse. We also conceive the ellipse cut by three other planes, of which two are vertical and the third horizontal. Let the less axis of the ellipse DE and its parallel MN be the intersections of the two first with the plane

ADBE, and *ST* that of the third with the same plane *ADBE*. This done, we say that the section of the ellipsoid by the plane represented by *ST* is an ellipse similar to *ADBE*.

We conceive perpendiculars to the plane *ADBE* to be raised from the points *O* and *R*, meeting the surface of the ellipsoid. These perpendiculars will be at once ordinates of the section made by *ST*, and of the circular sections made by *MN* and *DE*. Now since they are ordinates of the circular sections, if we call the perpendicular at the point R, z, and the perpendicular at the point O, t, we shall have

$$z^2 = DR \,.\, RE,$$
$$t^2 = MO \,.\, ON.$$

But, calling CD, $\frac{1}{2}\, b$; PM, y; CA, $\frac{1}{2}\, a$; and $CR = OP$, u; we have

$$DR = \tfrac{1}{2} b + u, RE = \tfrac{1}{2} b - u, MO = y + u, \; ON = y - u,$$

so that

$$DR \,.\, RE = \tfrac{1}{4}\, b^2 - u^2 = z^2,$$
$$MO \,.\, ON = y^2 - u^2 = t^2.$$

But, by the nature of the ellipse (*Ap.* 123), we have

$$y^2 = \frac{b^2}{a^2}\,(\tfrac{1}{4}\, a^2 - x^2),$$

calling CP, x. And k representing the ordinate SR to the less axis, we have (*Ap.* 122) $k^2 = \frac{a^2}{b^2}\,(\frac{1}{4}\, b^2 - u^2)$, whence we deduce $u^2 = \frac{1}{4}\, b^2 - \frac{b^2 k^2}{a^2}$; substituting these values of u^2 and y^2 in those of z^2 and t^2, we have

$$z^2 = \frac{b^2 k^2}{a^2}, \text{ and } t^2 = \frac{b^2 k^2}{a^2} - \frac{b^2 x^2}{a^2},$$

whence it is evident that

$$z^2 : t^2 :: \frac{b^2 k^2}{a^2} : \frac{b^2 k^2}{a^2} - \frac{b^2 x^2}{a^2} :: k^2 : k^2 - x^2 :: SR^2$$
$$\text{or } SR \,.\, RT : SO \,.\, OT;$$

that is, the square of the ordinate z, corresponding to the point R, is to the square of the ordinate t, corresponding to the point O, as the product of the two abscissas to the first, is to the product of the abscissas to the second; the section made by *ST* is therefore an ellipse.

Besides, the equation $z^2 = \frac{b^2 k^2}{a^2}$, or $z = \frac{b\,k}{a}$, gives

$$z : k :: b : a;$$

now z, or the ordinate to the point R, is the semi-minor axis of this ellipse, and k or SR is the semi-major axis; the two axes of this ellipse have therefore the same ratio as those of the generating ellipse; and as nothing in the course of this reasoning determines at what distance CR this section is supposed to be made, the same thing takes place for every section parallel to AB.

This being determined, if we wish to have the solidity of any segment of an ellipsoid, comprehended between the two parallel planes represented by AB and ST, we represent by S the surface of the generating ellipse; and since the ellipse, of which ST is the greater axis, is similar to this, we shall have the surface of this last by the proportion

$$\tfrac{1}{4}\,a^2 : k^2 :: S : \frac{S\,k^2}{\tfrac{1}{4}\,a^2};$$

multiplying this surface by the infinitely small thickness $R\,r$ or $d\,u$ of the elementary segment, we have $\dfrac{S\,k^2\,d\,u}{\tfrac{1}{4}\,a^2}$ for the value of this segment; but, according to what has just been said above, we have

$$k^2 = \frac{a^2}{b^2}\left(\tfrac{1}{4}\,b^2 - u^2\right);$$

whence the elementary segment will be

$$\frac{S\,d\,u\,(\tfrac{1}{4}\,b^2 - u^2}{\tfrac{1}{4}\,b^2}, = \frac{S}{\tfrac{1}{4}\,b^2}\,(\tfrac{1}{4}\,b^2\,d\,u - u^2\,d\,u),$$

of which the integral is $\dfrac{S}{\tfrac{1}{4}\,b^2}\,(\tfrac{1}{4}\,b^2\,u - \tfrac{1}{3}\,u^3)$, if we reckon from the centre C. But if we reckon from the point K, the integral will be

$$\frac{S}{\tfrac{1}{4}\,b^2}\,(\tfrac{1}{4}\,b^2\,u - \tfrac{1}{3}\,u^3) + C.$$

In order to determine C, we call CK, e; then the integral must be zero at the point K where $u = e$; we shall have, therefore,

$$\frac{S}{\tfrac{1}{4}\,b^2}\,(\tfrac{1}{4}\,b^2\,e - \tfrac{1}{3}\,e^3) + C = 0,$$

and consequently

$$C = -\frac{S}{\tfrac{1}{4}\,b^2}\,(\tfrac{1}{4}\,b^2\,e - \tfrac{1}{3}\,e^3),$$

wherefore every segment of an elongated ellipsoid, comprehended between two planes parallel to the greater axis, has for its expression

$$\frac{S}{\tfrac{1}{4}\,b^2}\,(\tfrac{1}{4}\,b^2\,u - \tfrac{1}{3}\,u^2) - \frac{S}{\tfrac{1}{4}\,b^2}\,(\tfrac{1}{4}\,b^2\,e - \tfrac{1}{3}\,e^3),$$

or $$\frac{S}{\frac{1}{4}b^2}\left(\tfrac{1}{4}b^2 u - \tfrac{1}{4}b^2 e - \tfrac{1}{3}u^3 + \tfrac{1}{3}e^3\right).$$

Now $$\tfrac{1}{4}b^2 u - \tfrac{1}{4}b^2 e = \tfrac{1}{4}b^2(u-e).$$

In like manner,

$$\tfrac{1}{3}e^3 - \tfrac{1}{3}u^3 = \frac{e-u}{3}(e^2 + eu + u^2).$$

Moreover, $u-e$ represents the distance of the two parallel planes or the altitude of the segment comprehended between them; if, therefore, we make $u-e=h$, calling this altitude h, and substitute for e its value $u-h$ drawn from this equation, we have, after all reductions, $$\frac{S}{\frac{1}{4}b^2}\left(\tfrac{1}{4}b^2 h - h u^2 + h^2 u - \frac{h^3}{3}\right),$$

or $$\frac{Sh}{\frac{1}{4}b^2}\left(\tfrac{1}{4}b^2 - u^2\right) + \frac{Sh^2}{\frac{1}{4}b^2}\left(u - \frac{h}{3}\right);$$

but we have had, above, $k^2 = \frac{a^2}{b^2}\left(\tfrac{1}{4}b^2 - u^2\right)$, and consequently

$$\tfrac{1}{4}b^2 - u^2 = \frac{b^2 k^2}{a^2}.$$

The value of the solid segment is therefore changed into

$$\frac{Shk^2}{\frac{1}{4}a^2} + \frac{Sh^2}{\frac{1}{4}b^2}\left(u - \frac{h}{3}\right).$$

But we found $\frac{Skk}{\frac{1}{4}a^2}$ for the expression of the surface of the section made by ST, or the inferior section; calling then this surface s, we shall have

$$sh + \frac{Sh^2}{\frac{1}{4}b^2}\left(u - \frac{h}{3}\right);$$

finally, if we represent by s' the surface of the section made by LK, and by l its semi-minor axis, we shall have, from the similarity of the sections, $\tfrac{1}{4}b^2 : l^2 :: S : s'$; whence $S = \tfrac{1}{4}\frac{b^2 s'}{l^2}$; which gives, for a final expression,

$$sh + \frac{s'h^2}{l^2}\left(u - \frac{h}{3}\right).$$

That is, we must, 1st, *Multiply the surface of the less section by the altitude of the segment*; 2d, *multiply the surface of the greater section by the ratio* $\frac{h^2}{l^2}$ *of the square of the altitude of the segment to the square of the semi-minor axis of the superior section,*

and by the distance from the centre to the inferior section minus a third of the altitude of the segment.

This rule may be usefully applied to measuring the solidity of that part of the hull of a ship which the lading causes to sink below the surface, whenever the figure of this part may be compared to a portion of an ellipsoid. In this case, s will represent the section of the vessel without its lading, at the surface of the water; s' the section with the lading; h the distance of the two sections; l the greatest breadth of s', and u the distance from the greatest horizontal section of the spheroid, to s.

As to the mode of measuring s and s', it may be observed, that one of these surfaces is determined by the other, since, belonging to similar ellipses, they must be to each other as the squares of their greater or less axes. It is therefore only necessary to know how to determine one of them. Now, we shall presently see, that the surface of an ellipse is to the surface of a circle which has for its diameter the greater axis of the ellipse, as the less axis is to the greater. As we know then how to estimate the surface of a circle, at least to as great exactness as may be desirable, we shall easily determine that of an ellipse whose axes are known.

On the integration of quantities containing Sines and Cosines.

108. We found (21, 22) that

$$d(\sin z) = dz \cos z,$$
$$d(\cos z) = -dz \sin z;$$

therefore, reciprocally,

$$\int dz \cos z = \sin z + C,$$
$$\int -dz \sin z = \cos z + C.$$

It is required to find the integral of $dz \cos 3z$; we have

$$\int dz \cos 3z = \int \frac{3\,dz \cos 3z}{3} = \frac{\sin 3z}{3} + C$$

In like manner

$$\int dz \sin 3z = \int \frac{-3\,dz \sin 3z}{-3} = \frac{\cos 3z}{-3} + C.$$

In general, m being any constant quantity,

$$\int dz \sin mz = \int \frac{-m\,dz \sin mz}{-m} = \frac{\cos mz}{-m} + C.$$

Let it be proposed to integrate $(\sin z)^n dz \cos z$. Because

$$(\sin z)^n dz \cos z = (\sin z)^n d(\sin z),$$

we have

$$\int (\sin z)^n\, dz \cos z = \frac{(\sin z)^{n+1}}{n+1} + C.$$

If the proposed differential were $(\sin m z)^n\, dz \cos m z$, we should give it the form

$$\frac{(\sin m z)^n\, m\, dz \cos m z}{m} = \frac{(\sin m z)^n\, d(\sin m z)}{m},$$

of which the integral is

$$\frac{(\sin m z)^{n+1}}{m(n+1)} + C.$$

In like manner

$$\int (\cos m z)^n\, dz \sin m z = \int \frac{(\cos m z)^n \times -m\, dz \sin m z}{-m}$$

$$= \frac{(\cos m z)^{n+1}}{-m(n+1)} + C.$$

Let it be proposed to integrate $dz \sin p z \cos q z$, p and q being constant quantities. By what has been demonstrated (*Tr.* 27),

$$\sin p z \cos q z = \tfrac{1}{2} \sin (p z + q z) + \tfrac{1}{2} \sin (p z - q z)$$
$$= \tfrac{1}{2} \sin (p+q) z + \tfrac{1}{2} \sin (p-q) z;$$

we have therefore to integrate

$$\tfrac{1}{2} dz \sin (p+q) z + \tfrac{1}{2} dz \sin (p-q) z$$
$$= \tfrac{1}{2} \frac{(p+q)\, dz \sin (p+q) z}{p+q} + \tfrac{1}{2} \frac{(p-q)\, dz \sin (p-q) z}{p-q}$$

of which the integral is

$$\frac{-\tfrac{1}{2} \cos (p+q) z}{p+q} - \frac{\tfrac{1}{2} \cos (p-q) z}{p-q} + C.$$

We should integrate in the same manner $dz \sin p z \cos q z \sin r z$, by converting these products into the sines or cosines of the sum or difference of the arcs $p z$, $q z$, $r z$, &c. (*Tr.* 27).

If we wished to integrate $dz (\sin z)^3$, we should change this differential into $dz \sin z (\sin z)^2$; now

$$\sin z^2 = \sin z \sin z = \tfrac{1}{2} \cos (z - z) - \tfrac{1}{2} \cos (z + z)$$
$$= \tfrac{1}{2} \cos o - \tfrac{1}{2} \cos 2 z = \tfrac{1}{2} - \tfrac{1}{2} \cos 2 z;$$

therefore

$$dz \sin z^3 = \tfrac{1}{2} \sin z\, dz - \tfrac{1}{2} dz \sin z \cos 2 z.$$

We should then resolve $\sin z \cos 2 z$ in the same way that $\sin p z \cos q z$ was resolved above, and the integration would be easy. We see, therefore, how we might integrate $dz \sin z^n$, n being any positive whole number. We should proceed in a similar manner to integrate $dz \cos z^n$. We may therefore, on the same principles, integrate quantities of the form

$$dz \sin p z^m \cos q z^n \sin r z^s, \text{ \&c.}$$

m, n, s being positive whole numbers.

Finally, these principles, what has been already shown (*Tr.* 27, &c.),

and what has been laid down above on the integration of quantities, give us the means of integrating differentials affected by sines and cosines, whenever they have an algebraical integral; and when tangents occur, they may be reduced to the differentials of sines and cosines, by observing that $\text{tang } z = \frac{\sin z}{\cos z}$.

On the mode of integrating by approximation, and some uses of that method.

109. This has nothing to do with simple differentials, since they, as we have already seen, are always easily integrated. It is only for complex differentials which elude the methods already given.

The art of integrating by approximation, consists in converting the proposed quantity into a series of simple quantities whose value continually diminishes; each term is then easily integrated, and it is sufficient to take a certain number of them, in order to obtain an opproximate value for the integral.

The rule given (*Alg.* 141) for raising a quantity to any proposed power, and which is equally applicable to polynomials, is the method we employ in order to integrate by approximation. We shall now give some examples.

110. Let it be proposed to find the length of the arc of a circle, AM (*fig.*40) by means of its versed sine AP.

Supposing the arc Mm to be infinitely small, if we draw Mr parallel to AP, and also the radius CM; the similar triangles CPM, Mrm, give

$$PM : CM :: Mr : Mm.$$

Making AP, x, and the diameter AB, 1, we have $Mr = dx$, $CM = \frac{1}{2}$, and PM† $= \sqrt{x - x^2}$. Therefore

$$\sqrt{x-x^2} : \tfrac{1}{2} :: dx : Mm = \frac{\frac{1}{2}dx}{\sqrt{x-x^2}},$$

and consequently $AM = \int \frac{\frac{1}{2}dx}{\sqrt{x-x^2}}$. This quantity cannot be integrated by the rules given above, wherefore we change it into $\int \frac{\frac{1}{2}dx}{x^{\frac{1}{2}}\sqrt{1-x}}$ (*Alg.* 123), and then into $\int \frac{1}{2}x^{-\frac{1}{2}}dx(1-x)^{-\frac{1}{2}}$

† $PM = \sqrt{\overline{MC}^2 - \overline{CP}^2} = \sqrt{\frac{1}{4} - (\frac{1}{2} - x)^2} = \sqrt{x - x^2}$.

(*Alg.* 133). We then reduce $(1-x)^{-\frac{1}{2}}$ to a series (*Alg.* 144); and find, after making all the reductions,

$$(1-x)^{-\frac{1}{2}}=1+\tfrac{1}{2}x+\tfrac{3}{8}x^2+\tfrac{5}{16}x^3+\&c.$$

therefore

$$\int\tfrac{1}{2}x^{-\frac{1}{2}}dx\,(1-x)^{-\frac{1}{2}}=\int\tfrac{1}{2}x^{-\frac{1}{2}}dx\,(1+\tfrac{1}{2}x+\tfrac{3}{8}x^2+\tfrac{5}{16}x^3+\&c.)$$

$$=\int(\tfrac{1}{2}x^{-\frac{1}{2}}dx+\tfrac{1}{4}x^{\frac{1}{2}}dx+\tfrac{3}{16}x^{\frac{3}{2}}dx+\tfrac{5}{32}x^{\frac{5}{2}}dx+\&c.)$$

$$=\frac{\frac{1}{2}x^{\frac{1}{2}}}{\frac{1}{2}}+\frac{\frac{1}{4}x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{\frac{3}{16}x^{\frac{5}{2}}}{\frac{5}{2}}+\frac{\frac{5}{32}x^{\frac{7}{2}}}{\frac{7}{2}}+\&c.$$

$$=x^{\frac{1}{2}}+\tfrac{1}{6}x^{\frac{3}{2}}+\tfrac{3}{40}x^{\frac{5}{2}}+\tfrac{5}{112}x^{\frac{7}{2}}+\&c.$$

a quantity to which there is no constant quantity to be added, because when $x=0$, it becomes zero, as should be the case, since then the arc AM, which it expresses, is zero.

By reason of the common factor $x^{\frac{1}{2}}$, we may give to the series expressing the arc AM the form

$$x^{\frac{1}{2}}(1+\tfrac{1}{6}x+\tfrac{3}{40}x^2+\tfrac{5}{112}x^3+\&c.)$$

We now observe that the versed sine x is always less than the diameter 1, except when we speak of the semicircumference, wherefore x is always a fraction, and consequently the values of the terms of the series will diminish, in proportion as the versed sine of the arc in question diminishes. If, therefore, we wished to find, for example, the length of the arc whose versed sine is the hundredth part of the diameter, we should have $x=\frac{1}{100}=0{,}01$, and consequently $x^{\frac{1}{2}}=0{,}1$; we should have therefore for the value of this arc

$$0{,}1\left(1+\frac{0{,}01}{6}+\frac{3\,(0{,}01)^2}{40}+\frac{5}{112}(0{,}01)^3\right);$$

and as the next successive term of this series would be at least a hundred times less than the last of those given, since each is more than a hundred times less than the preceding, if we ascertain what is the value of the term $\frac{5}{112}(0{,}01)^3$, we may, by taking the hundredth part of this value, judge of the degree of exactness to which we shall have the value of the arc, if we confine ourselves to these four first terms. Now $\frac{5}{112}(0{,}01)^3$ is

$$\tfrac{5}{112}(0{,}000001)=\frac{0{,}000005}{112}\quad 0{,}0000000446,$$

of which the hundredth part is 0,000 000 000 446; we may therefore, with certainty, estimate each term of our series as far as to ten decimals, without fearing that the value of the resulting arc should be faulty in the ninth decimal. Thus we shall have

$$\tfrac{5}{112}(0.01)^3 = 0{,}000\ 000\ 0446;\ \tfrac{3}{40}(0{,}01)^2 = 0{,}0000075000;$$

$$\frac{0{,}01}{6} = 0{,}0016666666;$$

the sum of the series will therefore be

$$0{,}1\ (1{,}0016742112), \text{ or } 0{,}100167421,$$

confining ourselves to 9 decimals, and we might with perfect safety admit even the tenth.

Such is the value of the arc whose versed sine is the hundredth part of the diameter. If, therefore, we knew how many times the number of degrees of this arc is contained in 360°, we should, by multiplying their length by the number of times, have the approximate value of the circumference. But this we do not know.

As we know (*Trig.* 18) that the sine of 30° is half the radius, and as, knowing the sine of an arc, we may easily find its versed sine, we might calculate the versed sine of 30°, substitute it for x in the above series, and then, multiplying the result by 12, which is the number of times that 30° is contained in 360°, we should have the approximate length of the circumference. But as the series would be little convergent, so that we should have to calculate a great number of terms, in order to obtain an approximate value of the circumference, we shall point out another way, which will serve as a second example of the method of approximation.

Draw the tangent AN (*fig.* 46) the secant CMN and the secant Cmn infinitely near to it; from the centre C, and with the radius CN, describe the infinitely small arc Nr, which may be considered as perpendicular to Cn. The small triangle Nrn will be similar to the right-angle triangle CAn, because, besides the right angle, they have a common angle n; it will be also similar to the triangle CAN, which differs infinitely little from CAn; we have therefore

$$CN : CA :: Nn : Nr = \frac{CA \times Nn}{CN};$$

now the similar sectors CNr, CMm, give

$$CN : CM, \text{ or } CA :: Nr = \frac{CA \,.\, Nn}{CN} : Mm = \frac{CA^2 \,.\, Nn}{\overline{CN}^2}.$$

Calling, therefore, AN, x; the radius CA, a; we have

$$Nn = dx \text{ and } CN = \sqrt{a^2 + x^2};$$

the value of Mm will thus become

$$\frac{a^2\,dx}{a^2 + x^2}; \text{ that is, } Mm = \frac{a^2\,dx}{a^2 + x^2};$$

therefore

$$\int Mm = AM = \int \frac{a^2\,dx}{a^2 + x^2}.$$

This quantity cannot be exactly integrated. In order to integrate it by approximation, we must put it under the form $\int a^2\,dx\,(a^2 + x^2)^{-1}$; then, having found (*Alg.* 144) that

$$(a^2 + x^2)^{-1} = a^{-2}\left(1 - \frac{x^2}{a^2} + \frac{x^4}{a^4} - \frac{x^6}{a^6} + \frac{x^8}{a^8} - \&c.\right),$$

we shall have

$$\int a^2\,dx\,(a^2 + x^2)^{-1} = \int dx \left(1 - \frac{x^2}{a^2} + \frac{x^4}{a^4} - \frac{x^6}{a^6} + \frac{x^8}{a^8} - \&c.\right)$$

$$= \int \left(dx - \frac{x^2\,dx}{a^2} + \frac{x^4\,dx}{a^4} - \frac{x^6\,dx}{a^6} + \frac{x^8\,dx}{a^8} - \&c.\right)$$

$$= x - \frac{x^3}{3a^2} + \frac{x^5}{5a^4} - \frac{x^7}{7a^6} + \frac{x^9}{9a^8} - \&c.$$

$$= x\left(1 - \frac{x^2}{3a^2} + \frac{x^4}{5a^4} - \frac{x^6}{7a^6} + \frac{x^8}{9a^8} - \&c.\right).$$

It now remains to ascertain whether we know any arc, which, being contained a known number of times in the circumference, has a known tangent. Now the arc of 45° is such an arc, being contained 8 times in the circumference, and its tangent is equal to the radius; supposing, therefore $x = a$, we shall have for the length of the arc of 45°, the value of this series

$$a\,(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11}, \&c.)$$

But, as the terms of this series decrease very slowly, we must endeavour to decompose the arc of 45° into two other arcs, whose tangents shall be known. It is not important that the number of degrees in these arcs be known, provided they make 45°; when we shall have calculated their lengths by means of their tangents, we shall, by adding these lengths together, have that of the arc of 45°. As these arcs will be less than 45°, their tangents will be

less than the radius, and, consequently, the series will be more convergent and more easy to calculate.

What has been laid down (*Trig.* 27) furnishes us the means of finding two such arcs. For, we found that a and b being any two arcs, we have

$$\tang(a+b)=\frac{\tang a+\tang b}{1-\tang a \tang b},$$

supposing the radius $=1$. If, therefore, we suppose $a+b=45^\circ$, in which case $\tang(a+b)=1$, we shall have

$$\frac{\tang a+\tang b}{1-\tang a \tang b}=1,$$

an equation, from which, by the common rules we deduce

$$\tang b=\frac{1-\tang a}{1+\tang a}.$$

Taking then the tang $a=\frac{1}{2}$; we shall have

$$\tang b=\frac{1-\frac{1}{2}}{1+\frac{1}{2}}=\frac{1}{3}$$

We have therefore only to calculate, by means of the series above, the length of the arc whose tangent x is $\frac{a}{2}$ or one half of the radius, and the length of the arc whose tangent is $\frac{a}{3}$: these two lengths added together will give the length of the arc of 45°. Now, substituting successively $\frac{a}{2}$ and $\frac{a}{3}$ in place of x in the preceding series, we have

$$\frac{a}{2}\left(1-\frac{1}{3.2^2}+\frac{1}{5.2^4}-\frac{1}{7.2^6}+\frac{1}{9.2^8}-\frac{1}{11.2^{10}}+\frac{1}{13.2^{12}}-\&c.\right)$$
$$\frac{a}{3}\left(1-\frac{1}{3.3^2}+\frac{1}{5.3^4}-\frac{1}{7.3^6}+\frac{1}{9.3^8}-\frac{1}{11.3^{10}}+\frac{1}{13.3^{12}}-\&c.\right)$$

If we wish to have the value of each of these arcs expressed exactly as far as the ninth decimal, we must calculate the first 15 terms of the first, and only the first 10 terms of the second. Now this calculation is very easily made by observing that in the first, we may calculate the consecutive terms, by forming at first a series, each term of which shall be equal to the preceding multiplied by $\frac{1}{2^2}$ that is, shall be $\frac{1}{4}$ of the preceding; we then multiply this series, term by term, by the series $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}$, &c.

finally, adding the odd terms together, and also the even terms together, and subtracting the sum of the former from the sum of the latter, we multiply the remainder by $\frac{a}{2}$. In like manner, the calculation of the second is reduced to forming a series, each term of which shall be equal to the preceding multiplied by $\frac{1}{3^2}$, that is, shall be $\frac{1}{9}$ of the preceding, and then proceeding as above, only that the result is to be multiplied by $\frac{a}{3}$ instead of $\frac{a}{2}$. If this operation is executed, and the approximation carried as far as 10 decimals, we shall have for the first series $\frac{a}{2}$ (0.9272952180), or a (0.4636476090); and for the second, $\frac{a}{3}$ (0.9652516632), or a (0.3217505544); therefore the arc of 45°, which is the sum of these two, will be a (0.7853981634). Taking, therefore, the quadruple, in order to obtain the semi-circumference, we shall have a (3.1415926536); therefore the radius is to the semi-circumference, or the diameter is to the circumference

$$: : a : a\ (3.1415926536) : : 1 : 3.1415926536,$$

a ratio which does not differ from that obtained in Geometry (*art.* 294), and which may be easily carried to still greater exactness.

111. As a third example of approximation, let it be proposed to find the logarithm of any number. But we must first observe what has been already mentioned (26), viz. that the logarithms here spoken of are not those found in the tables. But the former being calculated, we may immediately deduce from them the latter, as will be seen as soon as we have shown how to calculate the former.

We conceive the proposed number to be divided into two parts, represented by $a+x$; a being the greater part. According to what has been shown (26), we have

$$d \,.\, \log\,(a+x) = \frac{dx}{a+x},$$

a quantity which cannot be integrated algebraically. It must therefore be reduced to a series, and to this end put under the form . . $dx\,(a+x)^{-1}$. Now we have (*Alg.* 144.)

$$(a+x)^{-1}=a^{-1}\left(1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+\&c.\right)$$
$$=\frac{1}{a}-\frac{x}{a^2}+\frac{x^2}{a^3}-\frac{x^3}{a^4}+\&c.;$$

whence

$$d\,l\,(a+x)=dx\,(a+x)^{-1}$$
$$=\left(\frac{dx}{a}-\frac{x\,dx}{a^2}+\frac{x^2\,dx}{a^3}-\frac{x^3\,dx}{a^4}+\&c.\right);$$

integrating, we have

$$l\,(a+x)=\left(\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\frac{x^4}{4a^4}+\&c.\right)+C.$$

In order to determine the constant C, we observe that as this equation is universally true, it must be so when $x=0$, in which case it is reduced to $l\,a=C$; therefore $C=l\,a$, wherefore

$$l\,(a+x)=l\,a+\left(\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\frac{x^4}{4a^4}+\&c.\right)$$

Knowing, therefore, the logarithm of a single number, we may, by this series calculate the logarithm of any other number. If, for example, we suppose $a=10$, and $a+x=11$; we shall have $x=1$, and consequently $\frac{x}{a}=\frac{1}{10}$, whence we shall find

$$l\,11=l\,10+\left(0{,}1-\frac{(0{,}1)^2}{2}+\frac{(0{,}1)^3}{3}-\&c.\right.$$

which shows what must be added to the logarithm of 10 to give that of 11.

But as the general series just found is not sufficiently convergent, we may proceed in the manner following. Let it be proposed to find the logarithm of a fraction whose numerator is greater than its denominator; we shall presently see that the investigation of the logarithm of any number may always be referred to this case.

Let a represent the sum of the numerator and denominator of this fraction, and x their difference; then (*Alg.* 3) we have $\frac{1}{2}a+\frac{1}{2}x$ for the numerator, and $\frac{1}{2}a-\frac{1}{2}x$ for the denominator; and consequently $\frac{\frac{1}{2}a+\frac{1}{2}x}{\frac{1}{2}a-\frac{1}{2}x}$ for the fraction; or, suppressing the common factor $\frac{1}{2}$, the fraction will be represented by $\frac{a+x}{a-x}$, and consequently $l\,\frac{a+x}{a-x}=l\,(a+x)-l\,(a-x)$ will represent its

logarithm. If now we differentiate, considering a as constant, and x alone as variable,† we shall have (26)

$$\frac{dx}{a+x}+\frac{dx}{a-x}=\frac{2\,a\,dx}{a^2-x^2}=2\,a\,d\,x\,(a^2-x^2)^{-1};$$

reducing, therefore, $(a^2-x^2)^{-1}$ to a series, (*Alg.* 144), we have

$$(a^2-x^2)^{-1}=a^{-2}\left(1+\frac{x^2}{a^2}+\frac{x^4}{a^4}+\frac{x^6}{a^6}+\frac{x^8}{a^8}+\text{\&c.}\right);$$

wherefore

$$2\,a\,dx\,(a^2-x^2)^{-1}=2\,a^{-1}\,d\,x\left(1+\frac{x^2}{a^2}+\frac{x^4}{a^4}+\frac{x^6}{a^6}+\frac{x^8}{a^8}+\text{\&c.}\right)$$

$$=2\left(\frac{dx}{a}+\frac{x^2\,dx}{a^3}+\frac{x^4\,dx}{a^5}+\frac{x^6\,dx}{a^7}+\frac{x^8\,dx}{a^9}+\text{\&c.}\right).$$

Therefore

$$\int\frac{2\,a\,dx}{a^2-x^2}=\log\left(\frac{a+x}{a-x}\right)$$

$$=2\left(\frac{x}{a}+\frac{x^3}{3a^3}+\frac{x^5}{5\,a^5}+\frac{x^7}{7\,a^7}+\frac{x^9}{9a^9}+\text{\&c.}\right)+C.$$

The value of the constant C may be determined as above, by examining what the equation becomes when $x=0$. Now it is then reduced to $l\frac{a}{a}=C$; therefore $C=l\frac{a}{a}=l\,1=0$; we have therefore merely

$$l\,\frac{a+x}{a-x}=2\left(\frac{x}{a}+\frac{x^3}{3\,a^3}+\frac{x^5}{5\,a^5}+\frac{x^7}{7\,a^7}+\frac{x^9}{9\,a^9}+\text{\&c.}\right)$$

in which we see that each term is formed from the preceding by multiplying by the square of $\frac{x}{a}$ or of the first term; we then take the first term, $\frac{1}{3}$ of the second, $\frac{1}{5}$ of the third, &c., and double the sum.

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† Though this fraction represents any proposed fraction, this does not hinder our considering the sum a of the numerator and denominator as constant, because there is no fraction which may not be so prepared as to render the sum of the numerator and denominator equal to any number whatever. For example, to bring the fraction $\frac{3}{5}$ to such a form as to have 12 for the sum of its numerator and denominator, we have only to multiply the two terms by u, which gives $\frac{3\,u}{5\,u}$, and make $3\,u+5\,u=12$, or $8\,u=12$, whence we have $u=\frac{12}{8}=\frac{3}{2}$; then $\frac{3}{5}=\frac{\frac{9}{2}}{\frac{15}{2}}$, of which the sum of the numerator and denominator is, in fact, 12.

Let us apply it to some examples. Suppose, for instance, it were required to find the logarithm of 2. In order to this we represent 2 under the form of $\frac{3}{1}$; we shall then have $a = 3$, and $x = 1$; wherefore $\frac{x}{a} = \frac{1}{3}$, and $\frac{x^2}{a^2} = \frac{1}{9}$. Each term may then be easily formed, as we have only to take $\frac{1}{9}$ of the preceding term to form the series $\frac{x}{a}$, $\frac{x^3}{a^3}$, $\frac{x^5}{a^5}$ &c. thus we shall have,

$\frac{x}{a}$	$= 0{,}333333333$ . . Therefore	$\frac{x}{a}$	$= 0{,}333333333$
$\frac{x^3}{a^3}$	$= 0{,}037037037$	$\frac{x^3}{3a^3}$	$= 0{,}012345679$
$\frac{x^5}{a^5}$	$= 0{,}004115226$	$\frac{x^5}{5a^5}$	$= 0{,}000823045$
$\frac{x^7}{a^7}$	$= 0{,}000457247$	$\frac{x^7}{7a^7}$	$= 0{,}000065321$
$\frac{x^9}{a^9}$	$= 0{,}000050805$	$\frac{x^9}{9a^9}$	$= 0{,}000005645$
$\frac{x^{11}}{a^{11}}$	$= 0{,}000005645$	$\frac{x^{11}}{11a^{11}}$	$= 0{,}000000513$
$\frac{x^{13}}{a^{13}}$	$= 0{,}000000627$	$\frac{x^{13}}{13a^{13}}$	$= 0{,}000000048$
$\frac{x^{15}}{a^{15}}$	$= 0{,}000000069$	$\frac{x^{15}}{15a^{15}}$	$= 0{,}000000004$
	of which the sum is		0,346573588

and double this, or the log. 2, is 0.693147176, which, confining ourselves to 8 decimals (for, to answer for the accuracy of the ninth, we ought to have carried the approximation farther) is 0,69314718.

Since 4 is the square of 2, and 8 is its cube, double this logarithm will be the logarithm of 4, and the triple of it will be the logarithm of 8.

In order to obtain that of 3, we may calculate, in the same way, the logarithm of the fraction $\frac{4}{3}$, which being taken from that of 4, will give the logarithm of 3, since 3 is 4 divided by $\frac{4}{3}$, therefore $l\,3 = l\,4 - l\,\frac{4}{3}$; but it may be found more easily by calculating the logarithm of the fraction $\frac{8}{9}$, and subtracting it from the logarithm of 8, which has already been found; the remainder will be the logarithm of 9, half of which will be the logarithm of

3. Adding that of 3 to that of 2, we shall have the logarithm of 6. In order to obtain the logarithm of 5, we must first find that of 10, by calculating that of $\frac{10}{8}$, which, being added to the logarithm of 8, will give that of 10. By subtracting from this last the logarithm of 2, we obtain the logarithm of 5.

We thus see what is to be done in order to calculate any other logarithm. But it ought to be observed, that the calculation becomes shorter and shorter in proportion as the number becomes greater, so that when we have the logarithms only as far as 10, we may calculate the others as far as 100 without employing more than three terms of the series, if we confine ourselves to 8 decimals; beyond 100, the two first terms are sufficient, until we get to 1,000, and above that, a single term is sufficient.

In order to reduce these logarithms to those of the common tables, we must previously have the logarithm of 10. Now, if we calculate the logarithm of $\frac{10}{8}$, by the preceding formula, we shall find $l\,\frac{10}{8} = 0{,}22314355$; adding to this the logarithm of 8, which is found by tripling the logarithm of 2 obtained above, we have $l\,10 = 2{,}30258509$.

112. Having done this, we observe that the equation

$$d\,x = \frac{d\,y}{y},$$

on which (26) is founded the present calculation of logarithms, agrees only to the system of logarithms in which we suppose the modulus $= 1$; but that the equation which applies to all possible systems of logarithms, is

$$d\,x = \frac{m\,a\,d\,y}{y};$$

and that which applies to all the systems of logarithms in which we suppose that the first term a of the fundamental geometrical progression is 1, is

$$d\,x = \frac{m\,d\,y}{y}.$$

The first, viz. $d\,x = \frac{d\,y}{y}$, gives, on being integrated, $x = l\,y$; and the second, viz. $d\,x = \frac{m\,d\,y}{y}$, gives $x = m\,l\,y$, which shows, since x represents the logarithm, that to reduce the logarithms resulting immediately from calculation to those of any other system whose modulus is m, we must multiply them by m. Now the

logarithm of 10, in the common tables, is 1; and we have just seen that the logarithm of 10, which is given by the calculation, is 2,30258509; we have therefore $m \times 2,30258509 = 1$; wherefore the modulus of the common tables is $\frac{1}{2,30258509}$, which, by performing the division, is reduced to 0,43429448.

Therefore, *to reduce the logarithms given immediately by the calculus to the logarithms of the tables, we must multiply them by* 0,43429448. And, reciprocally, *to reduce the logarithms of the tables to those resulting immediately from the calculus, we must divide them by* 0,43429448, *or, which is more convenient and comes to the same thing, multiply them by* 2,30258509.

Thus, if 0,69314718, which was obtained above as the logarithm of 2, be multiplied by 0,43429448, it gives 0,3010300 for the logarithm of 2, which it is in fact in the common tables.

113. If we wish to go back from the logarithm to the number itself, we may proceed thus. We have seen above, that, representing any number by $a + x$, we had

$$l\,(a+x) = l\,a + \left(\frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4}\right) + \&c.$$

therefore

$$l\,(a+x) - l\,a = l\frac{a+x}{a} = \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \&c.$$

a being an arbitrary number, but such that its logarithm may differ little from that which is given, and which is supposed to be the logarithm of $a + x$. For the sake of greater simplicity, let

$$l\left(\frac{a+x}{a}\right) = z,$$

and we shall have

$$z = \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \&c.$$

It is required to obtain the value of $\frac{x}{a}$ in terms of z.

Let us suppose that this value may be expressed by

$$\frac{x}{a} = A\,z + B\,z^2 + C z^3 + D\,z^4 + \&c.$$

A, B, C, &c. being constant coefficients, which we wish to determine. We shall therefore have

$$z \text{ or } \left\{ \begin{array}{llllll} \frac{x}{a} & = A z & + B z^2 & + C z^3 & + D z^4 & + \text{\&c.} \\ -\frac{x^2}{2a^2} & = \ldots & -\frac{A^2}{2} z^2 & -\frac{2AB}{2} z^3 & -\frac{B^2}{2} z^4 & - \text{\&c.} \\ & \ldots & \ldots & \ldots & -\frac{2AC}{2} z^4 & - \text{\&c.} \\ +\frac{x^3}{3a^3} & = \ldots & \ldots & \frac{A^3}{3} z^3 & +\frac{3A^2B}{3} z^4 & + \text{\&c.} \\ -\frac{x^4}{4a^4} & = \ldots & \ldots & \ldots & -\frac{A^4}{4} z^4 & - \text{\&c.} \end{array} \right.$$

Now, in order that this equation may hold true, whatever be the value of z, it is necessary, 1st, that $A = 1$; 2d, that the sum of the terms which multiply each power of z in the other columns be zero. We have therefore,

$$B - \frac{A^2}{2} = 0; \quad C - AB + \frac{A^3}{3} = 0,$$

$$D - \frac{B^2}{2} - AC + A^2B - \frac{A^4}{4} = 0;$$

whence we deduce

$$B = \tfrac{1}{2} = \frac{1}{1.2}, \; C = \tfrac{1}{6} = \frac{1}{1.2.3}, \; D = \tfrac{1}{24} = \frac{1}{1.2.3.4};$$

and if we suppose a still greater number of terms in the series, such as $E z^5$, $F z^6$, &c. we should find, in like manner,

$$E = \frac{1}{1.2.3.4.5}, \; F = \frac{1}{1.2.3.4.5.6}, \text{ \&c.}$$

we have therefore

$$\frac{x}{a} = z + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} + \frac{z^5}{1.2.3.4.5} + \frac{z^6}{1.2.3.4.5.6} + \text{\&c.}$$

Therefore

$$1 + \frac{x}{a} = \frac{a+x}{a}$$

$$= 1 + \frac{z}{1} + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} + \frac{z^5}{1.2.3.4.5} + \text{\&c.}$$

To make use of this formula, we must take from the given logarithm, which is that of $a+x$, the nearest known logarithm, the number corresponding to which we take for a. Then we shall have $l\,\frac{a+x}{a}$, or z, which may be substituted in the preceding formula. The result will be the value of $\frac{a+x}{a}$; whence we may easily deduce the value of $a+x$, since a will be known.

As we speak here of the logarithms which have 1 for the modulus, if the logarithm given were of the nature of those in the common tables, we should have to begin by reducing it, and also that which

we took for the logarithm of a, or merely their difference, to these logarithms, which might be done by the methods given (112).

If we wish to know what is the number whose logarithm is 1, in the system here treated of, we must suppose $l\left(\frac{a+x}{a}\right)$, or $z=1$, and we shall have

$$\frac{a+x}{a}=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\frac{1}{1.2.3.4.5}+\&c.$$

which will be found $=2,7182818$, confining ourselves to 7 decimals.

The number whose logarithm is 1 is often met with in the calculus; we shall meet with it hereafter, and it is for that reason that we have shown the method of calculating it.

114. We may have another expression for a number by means of its logarithm. If, for example, x were the number given, and $l\,x=z$, we multiply the second member of this equation by $l\,e$, e being the number whose logarithm is 1, and have $l\,x=z\,l\,e$, which does not effect any change, since $l\,e=1$. Now the equation $l\,x=z\,l\,e$ is changed, by the nature of logarithms, into $l\,x=l\,e^z$; whence we deduce $x=e^z$, because, the logarithms being equal, the quantities to which they belong must be equal.

But, agreeably to what has been said in article 113, if we have $l\,x=z$, we have $x=1+z+\frac{z^2}{1.2}+\&c.$ And as we have at the same time, $x=e^z$, we shall have

$$e^z=1+z+\frac{z^2}{1.2}+\frac{z^3}{1.2.3}+\frac{z^4}{1.2.3.4}+\&c.$$

Remark.

115. The method which we have employed to deduce the value of x from the equation $z=\frac{x}{a}-\frac{x^2}{2a^2}+\&c.$ is called the *inverse method* of series. It consists, as we have seen, in supposing the variable, whose value we would find, to be expressed by a series, in which the other variable has exponents in arithmetical progression, and each term an indeterminate constant coefficient.

If we had several terms in x and z in the same equation, without having x and z multiplied together, we might determine the series of the exponents, by making the exponent of the first term of the supposed series, equal to the smallest exponent of the same variable in the given equation, and taking as common difference of the exponents of the same series, the greatest common divisor of the exponents of this same variable in the equation. If we had, for example,

$$z^{\frac{2}{3}}+3z=2x-\tfrac{2}{5}x^2+\tfrac{3}{7}x^3+\&c.,$$

we should make

$$x=A\,z^{\frac{2}{3}}+B\,z+Cz^{\frac{4}{3}}+Dz^{\frac{5}{3}}+Ez^2+\&c.$$

because the least exponent of z is $\frac{2}{3}$, and the greatest common divisor of the exponents $\frac{2}{3}$ and 1, of z, is $\frac{1}{3}$.

But if x and z were multiplied together, it would be necessary to pursue a method, the detail of which does not fall within the limits of our plan.

Uses of the preceding Approximations, in the integration of different quantities.

116. As there are tables already calculated of the different parts of the circle, as well as of logarithms, when we have to integrate any differential, which may be referred to the circle, or to logarithms, it will hereafter be unnecessary to reduce these differentials to series, as we may refer at once to those tables. We shall therefore now point out such of these differentials as most frequently occur, and show how the arcs of a circle, or the logarithms, which are their integral, may be determined.

117. We have seen (98) that $\frac{\frac{1}{2} a\, d x}{\sqrt{a x - x^2}}$ expresses the element of a circular arc AM (*fig.* 40), of which a is the diameter and x the abscissa; so that the integral of this quantity, or $\int \frac{\frac{1}{2} a\, d x}{\sqrt{a x - x^2}}$ is represented by the arc AM. Suppose, therefore, that it is required to find the value of this integral for a determinate value of x; from CA or $\frac{1}{2} a$, we subtract the known value of x or AP, and obtain CP. In the right-angled triangle CPM, we know the right angle, the hypothenuse $CM = \frac{1}{2} a$, and the side CP; we may therefore calculate the angle ACM; and knowing the angle ACM or the number of degrees in the arc AM, and also its radius CM, it is easy to calculate the length of this arc (*Geom.* 294).

118. If we had $\frac{h\, d x}{\sqrt{g k x - p x^2}}$, h, g, p, and k being known quantities, we should render this differential similar to the preceding, by dividing both terms of the fraction by $\sqrt{p}$, which would give

$$\frac{\frac{h}{\sqrt{p}}\, d x}{\sqrt{\frac{g k}{p} x - x^2}} = \frac{h}{\sqrt{p}} \cdot \frac{d x}{\sqrt{\frac{g k}{p} x - x^2}};$$

now if we had, as multiplier of $d x$, half of the quantity $\frac{g k}{p}$, by which x is multiplied in the radical, this differential would be similar to that of the preceding article; we therefore give it

$PpmM = dx\sqrt{ax-x^2}$; therefore every differential having this form, or susceptible of being reduced to this by preparations similar to those just pointed out, may be integrated by means of the half segment of a circle, whose abscissa is x and diameter a, a segment easily determined as well by the methods just laid down, as by those of elementary Geometry.

If, for example, we wish to find the surface of the elliptical half segment AMP (*fig.* 47,) we have $y = \frac{b}{a}\sqrt{ax-x^2}$; wherefore

$$y\,dx = d\,(APM) = \frac{b\,dx}{a}\,.\,\sqrt{ax-x^2}\,;$$

now $dx\sqrt{ax-x^2}$ expresses the element of the circular half segment APM', supposing a circle to be described upon AB, as a diameter; we have therefore

$$d\,(APM) = \frac{b}{a}\,d\,(APM'),$$

and integrating, $APM = \frac{b}{a}\,APM'$, which gives

$$APM : APM' :: b : a\,;$$

that is, the surface of the elliptical half segment, is to the surface of the corresponding circular half segment, as the less axis is to the greater axis; whence it is easy to conclude that the whole surface of the ellipse is to that of the circle described on its greater axis, as the less axis is to the greater; which we promised (107) to demonstrate.

If, instead of reckoning the abscissas from the point A, (*fig.*40), we reckon them from the centre C, then calling CA, $\frac{1}{2}a$, and CP, x; we shall have $-dx\sqrt{\frac{1}{4}a^2 - x^2}$ for the element of the half segment APM, because then $y = \sqrt{\frac{1}{4}a^2 - x^2}$, and the segment APM diminishes while x increases, which makes the differential of APM negative.

The following is an example of a differential, to be referred to this form.

122. Let it be proposed to find the surface of the elongated elliptical spheroid. The general formula for this kind of surfaces is $\frac{cy}{r}\sqrt{dx^2+dy^2}$ (98): now the equation of the ellipse is

$$y^2 = \frac{b^2}{a^2}\,(\tfrac{1}{4}\,a^2 - x^2)\,;$$

therefore

$$y = \frac{b}{a}\sqrt{\tfrac{1}{4}a^2 - x^2}, \text{ and } dy = -\frac{b}{a} \times \frac{x\,dx}{\sqrt{\tfrac{1}{4}a^2 - x^2}};$$

therefore

$$\frac{cy}{r}\sqrt{dx^2 + dy_2} = \frac{cb}{ra} \times \sqrt{\tfrac{1}{4}a^2 - x^2} \times \sqrt{dx^2 + \frac{b^2}{a^2} \times \frac{x^2\,dx^2}{\tfrac{1}{4}a^2 - x^2}},$$

or, performing the multiplication indicated, reducing, and raising $d\,x^2$ from under the radical,

$$\frac{cb\,dx}{ra}\sqrt{\tfrac{1}{4}a^2 - x^2 + \frac{b^2\,x^2}{a^2}}, \text{ or } \frac{cb\,dx}{ra}\sqrt{\frac{\tfrac{1}{4}a^4 - a^2x^2 + b^2x^2}{a^2}};$$

now, if we call the distance CF to the focus F, k, (*fig.* 48), we have $k^2 = \tfrac{1}{4}a^2 - \tfrac{1}{4}b^2$, or $4\,k^2 = a^2 - b^2$ (*Trig.* 112); therefore the element of the surface becomes

$$\frac{cb\,dx}{ra}\sqrt{\frac{\tfrac{1}{4}a^4 - 4\,k^2x^2}{a^2}} = \frac{cb\,dx}{ra^2}\sqrt{\tfrac{1}{4}a^4 - 4\,k^2x^2}.$$

Dividing under the radical by $4\,k^2$, and multiplying out of it by its root $2\,k$, and we have $\frac{2\,cbk\,dx}{ra^2} \cdot \sqrt{\frac{\tfrac{1}{16}a^4}{k^2} - x^2}$, a quantity to which we must give the sign $-$ to make it express the surface reckoned from the point A, because this surface diminishes in proportion as x increases; thus we have

$$-\frac{2\,cbk\,dx}{ra^2}\sqrt{\frac{\tfrac{1}{16}a^4}{k^2} - x^2}.$$

Comparing this with $-dx\sqrt{\tfrac{1}{4}a^2 - x^2}$, which we have just found to be the expression for a circular half segment, whose radius is $\tfrac{1}{2}a$, we shall conclude that the integral of $-dx\sqrt{\frac{\tfrac{1}{16}a^4}{k^2} - x^2}$, is a circular half segment $OM'P$, whose radius is $\frac{\tfrac{1}{4}a^2}{k}$, and whose abscissa taken from the centre is x, plus a constant quantity. Therefore, if with a radius $CO = \frac{\tfrac{1}{4}a^2}{k}$, that is, a third proportional to CF and CA, we describe the circle ONR, we shall have

$$\int -dx\sqrt{\frac{\tfrac{1}{16}a^4}{k^2} - x^2} = OPM' + C;$$

therefore

$$\int -\frac{2\,cbk\,dx}{ra^2}\sqrt{\frac{\tfrac{1}{16}a^4}{k^2} - x^2} = \frac{2\,cbk}{ra^2} \times OPM' + \frac{2\,cbk}{ra^2} \times C.$$

In order to determine the constant C, we must observe, that the surface sought, having its origin at the point A, must be zero at that point; but, at the point A, the half segment OPM' becomes OAN; we have, therefore,

$$0 = \frac{2cbk}{ra^2} \times OAN + \frac{2cbk}{ra^2} \times C;$$

whence we deduce $C = - OAN$; therefore the entire integral is $\frac{2cbk}{ra^2} \times OPM' - \frac{2cbk}{ra^2} \times OAN$, or $\frac{2cbk}{ra^2}(OPM' - OAN)$, or finally

$$\frac{2cbk}{ra^2}\ (APM'N).$$

Therefore the surface of the half spheroid will be

$$\frac{2cbk}{ra^2}(ACRN),$$

or, since $CO = \frac{\frac{1}{4}a^2}{k}$, and consequently $\frac{2k}{a^2} = \frac{1}{2CO}$, this surface will be $\frac{c}{r} \times \frac{b}{2CO} \times ACRN$, or $\frac{c}{r} \times \frac{CD}{CO} \times ACRN$; and that of the whole spheroid is double of this.

The mode of determining the radius CO is very simple. From the point C, as a centre, and with the radius CA, we describe the arc AL, cutting in L, the line FL perpendicular to CA from the point F; we produce CL until it meets at N the perpendicular AN drawn from the point A; this gives CN for the value sought of CO, or $\frac{\frac{1}{4}a^2}{k}$; for, the similar triangles CFL and CAN, give

$$CF : CL :: CA : CN,$$

or

$$k : \tfrac{1}{2}a :: \tfrac{1}{2}a : CN = \frac{\frac{1}{4}a^2}{k} = CO.$$

We may make use of the above result in measuring the surface of the hull of ships, which may be compared to the surface of an ellipsoid even more properly than their solidity can to the solidity of the same ellipsoid. The whole operation consists in calculating the angle ACN of the right-angled triangle ACN, of which we know two sides and the right angle. Thus the angle NCR becomes known, and we may easily deduce from it the surface of the sector NCR, to which adding that of the triangle CAN, we have $ACRN$, which we have only to multiply by $\frac{c}{r} \times \frac{CD}{CO}$. When we have found the surface of the hull, by multiplying it by the thick-

ness of the *sheathing*, we obtain the solid contents of the volume by which the hull is increased by sheathing.

123. With regard to the quantities which are immediately referred to logarithms, they are those in which the proposed differential is, or may be made to be, a fraction, whose numerator is the differential of the denominator, or this differential multiplied or divided by a constant number.

When the numerator is exactly the differential of the denominator, the integral is the logarithm of the denominator. Thus

$$\int\frac{dx}{x}=lx+C;\ \int\frac{dx}{a+x}=l(a+x)+C;$$

$$\int\frac{2x\,dx}{a^2+x^2}=l(a^2+x^2)+C.$$

But when the numerator is the differential of the denominator, multiplied or divided by a constant number, the proposed differential must be decomposed into two factors, of which one shall be a fraction having for its numerator the exact differential of the denominator, and the other a constant number. Then the integral will be the logarithm of the variable denominator, multiplied by the constant factor. For example, to integrate $\frac{ax^2\,dx}{a^3+x^3}$, as the differential of a^3+x^3 is $3x^2\,dx$, the differential must be so prepared as to have $3x^2\,dx$, in the numerator. To this end, we write it thus:

$$\frac{a}{3}\cdot\frac{3x^2\,dx}{a^3+x^3},\ \text{whose integral is}\ \frac{a}{3}\,l(a^3+x^3)+C.$$

In like manner

$$\int\frac{dx}{a-x}=\int\frac{1}{-1}\cdot\ \frac{-1\,dx}{a-x}=-l(a-x)+C$$

$$=0-l(a-x)+C=l\,1-l(a-x)+C=l\frac{1}{a-x}+C.$$

So

$$\int\frac{x\,dx}{a^2+x^2}=\tfrac{1}{2}\cdot\frac{2x\,dx}{a^2+x^2}=\tfrac{1}{2}\,l(a^2+x^2)+C=l\sqrt{a^2+x^2}+C.$$

Finally,

$$\int\frac{ax^{n-1}\,dx}{k+bx^n}=\int\frac{a}{bn}\cdot\frac{nbx^{n-1}dx}{k+bx^n}$$

$$=\frac{a}{bn}\,l(k+bx^n)+C=l(k+bx^n)^{\frac{a}{bn}}+C.$$

The following is an example of the manner of determining these integrals, in numbers. Suppose that it is required to find the value of $l\,(a+x)$, (a being 5,) when x is 2. It is then $l\,7$ which we wish to find. We take the logarithm of 7 from the common tables, which is 0,8450980; we then multiply it (112) by 2,30258509, or 2,3025851, and obtain 1,9459100 or 1,94591 for the value of $l\,(a+x)$ or of the integral of $\frac{dx}{a+x}$, when $a=5$ and $x=2$.

We sometimes meet with differentials which are integrated directly by logarithms, although they cannot be prepared like the preceding. For example, $\frac{dx}{\sqrt{x^2-1}}$ is of such a kind. We sometimes succeed in giving them the form of a logarithmic differential, by multiplying them by such a function of x that the product may become the differential of this function, or that differential multiplied or divided by a constant number. If then we divide by this function, the differential would evidently become a logarithmic differential. Applying this observation to $\frac{dx}{\sqrt{x^2-1}}$, we multiply it by $x+\sqrt{x^2-1}$, and we have $\frac{x\,dx}{\sqrt{x^2-1}}+dx$, which is the differential of $x+\sqrt{x^2-1}$; so that we have

$$\int\frac{dx}{\sqrt{x^2-1}}=\int\frac{dx+\frac{x\,dx}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}}=l\,(x+\sqrt{x^2-1})+C.$$

We shall also find the integral of $\frac{dx}{\sqrt{1-x^2}}$, by first multiplying both terms by $\sqrt{-1}$, which gives $\frac{dx\sqrt{-1}}{\sqrt{x^2-1}}$, whose integral, as we have just seen, is $\sqrt{-1}\;l\,(x+\sqrt{x^2-1})+C$.

124. In *art.* 82 we promised to explain how it happens that the fundamental rule for the integration of simple quantities gives an infinite quantity for the integral of $\frac{dx}{x}$, while this integral is expressed by $l\,x$, or at least $l\,x+C$.

The integral of $\frac{dx}{x}$ may be finite or infinite, according to the portion of it which we choose to take. To illustrate this, we first observe that to take the integral of $\frac{dx}{x}$, is nothing else than to

square the common hyperbola, considered with reference to its asymptotes. For the equation of this curve is

$$x y = a^2, \text{ or } x y = 1,$$

if we suppose for the sake of greater simplicity, that $a = 1$. Now, from this equation we deduce $y = \frac{1}{x}$; wherefore $y\,d\,x$, the element of the surface, becomes $\frac{d x}{x}$; if therefore we wish to have the space reckoned from the asymptote AZ (*fig.* 49), the integral of $\frac{d x}{x}$, or $l x + C$ must become zero, when the point P falls upon the point A, or when $x = 0$; in which case, therefore, we have $l\,0 + C = 0$, and consequently $C = -l\,0$; therefore the integral is $l x - l\,0$ or $l \frac{x}{0}$; that is, the space $ZAPMO$ reckoned from the asymptote is infinite, Z and V being considered as the extremities of the asymptote and of the corresponding branch of the hyperbola; in which there is nothing surprising.

But if, the point O being the vertex of the hyperbola, in which case the corresponding abscissa $AN = 1$, we wish to find the space reckoned from the point N, the integral $l x + C$ must become zero, when the point P shall fall upon the point N, or when $x = 1$; we have therefore $l\,1 + C = 0$, and consequently

$$C = -l\,1 = 0;$$

wherefore the space $NOMP$ is expressed by $l x$.

We see from this, 1st, that the logarithms immediately resulting from calculation, express the hyperbolic spaces comprehended between the asymptote of the curve, and reckoned from O the vertex of the curve; 2d, that if the integral $\frac{d x}{x}$ or $x^{-1}\,d\,x$ is infinite when found by the fundamental rule, it is because it expresses the space reckoned from the origin of the asymptotes.

125. As an application of integration by logarithms, let it be proposed to explain and apply the principles of the construction of *reduced Maps*.†

These maps have been invented in order to facilitate the laying down courses in navigation. The course, at least for a certain

† Maps on Mercator's projection.

time, is constantly on the same rhumb, and consequently makes constantly the same angle with each meridian which is successively crossed. Whence it follows, that if we wished to trace this course on a common map, in which all the meridians tend towards the same point, we should find it a curve line, and consequently very inconvenient for the operations which it is necessary to perform. It has been found better to represent the meridians by parallel straight lines, in order that the course which makes a constant angle with these meridians may be a straight line.

But, supposing that AMP, $am\,P$, (*fig.* 50), are two meridians; $A\,a$, a portion of the equator comprehended between these two meridians, and $M\,m$ the corresponding portion of any parallel; we see that the interval $M'\,m'$, (*fig.* 51), which represents the arc $M\,m$, is of the same magnitude as $A'\,a'$, which represents $A\,a$, the corresponding portion of the equator; although Mm is smaller than $A\,a$, in the proportion of $P'\,M$ to CA; which may be easily shown by drawing MP', $m\,P'$, AC, and $a\,C$, perpendicular to CP, for these lines form with the arcs $A\,a$ and Mm, the similar sectors $CA\,a$ and PMm.

To compensate for the increase given to $M\,m$, by representing it on the map by $M'\,m'$, we represent the latitude AM by a line $A'M'$, longer, when compared with $A'\,a'$ than AM compared with $A\,a$, in a certain ratio. And it is this ratio which it is now proposed to ascertain.

Let therefore M and R be two points infinitely near each other upon the meridian AM; $M\,m$, $R\,r$ the corresponding portions of two parallels. If we wish to represent Mm, $R\,r$ by the lines $M'\,m'$, $R'\,r'$, equal to the line $A'\,a'$, which represents $A\,a$, and still preserve the same ratio between the parts of each parallel, and those of the meridian, we must make the infinitely small interval $M'R'$, which separates $M'\,m'$ and $R'\,r'$, as much larger than MR as Mm is smaller than $A\,a$, that is, we must have

$M'R' : MR :: A\,a : Mm :: CA : P'M ::$ the radius is to the cosine of the latitude AM.

If, therefore, we call the latitude AM, s, the straight line $A'M'$, which represents it on the map, and which is called the *Increasing latitude*, will be indicated by s'. Take the radius CA or $CM=1$; CP' or the sine of the latitude $AM=x$; we shall have

$$P'M = \sqrt{1-x^2}\,;\ MR = ds = \frac{dx}{\sqrt{1-x^2}}\,; \text{ for}$$

$$P'M : CM :: tR : MR, \text{ or } \sqrt{1-x^2} : 1 :: dx : \frac{dx}{\sqrt{1-x^2}}.$$

But $$P'M : CA :: MR : M'R'\,;$$

therefore $$\sqrt{1-x^2} : 1 :: \frac{dx}{\sqrt{1-x^2}} : ds' = \frac{dx}{1-x^2}.$$

It is therefore necessary, in order to obtain s', to integrate the value of ds', that is, $\frac{dx}{1-x^2}$, Now, we have seen (111) that $\frac{2a\,dx}{a^2-x^2}$ resulted from the differentiation of $l\frac{a+x}{a-x}$; therefore $\frac{2dx}{1-x^2}$ results from that of $l\frac{1+x}{1-x}$; and consequently the integral of $\frac{dx}{1-x^2}$, which is one half of $\frac{2dx}{1-x^2}$, will be $\frac{1}{2}l\,\frac{1+x}{1-x}$, we shall therefore have

$$s' = \tfrac{1}{2}l\frac{1+x}{1-x} = l\sqrt{\frac{1+x}{1-x}}\,;$$

an integral to which there is no constant quantity to be added, because when $x = 0$, $l\sqrt{\frac{1+x}{1-x}}$ becomes $\log\sqrt{1} = l\,.\,1 = 0$; now s', or the straight line $A'M'$, must in fact be in that case zero, because the arc AM, which it represents, is zero when x is zero.

It may be now observed, in order to render this expression more convenient, that the radius is the sine of 90°; and since by x we have understood CP', or the sine of the latitude AM, we shall have, instead of $s' = l\sqrt{\frac{1+x}{1-x}}$, this equation,

$$s' = \log\sqrt{\frac{\sin 90^\circ + \sin AM}{\sin 90^\circ - \sin AM}}.$$

Now, (*Trig.* 28),

$$\frac{\sin 90^\circ + \sin AM}{\sin 90^\circ - \sin AM} = \frac{\operatorname{tang}(45^\circ + \frac{1}{2}AM)}{\operatorname{tang}(45^\circ - \frac{1}{2}AM)},$$

therefore

$$s' = \log\sqrt{\frac{\operatorname{tang}.(45^\circ + \frac{1}{2}AM)}{\operatorname{tang}.(45^\circ - \frac{1}{2}AM)}}.$$

But (*Trig.* 9), $(\operatorname{tang} 45^\circ - \frac{1}{2}AM) : 1 :: 1 : \cot(45^\circ - \frac{1}{2}AM)$.

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Moreover, $\cot(45^\circ - \frac{1}{2} AM) = \text{tang}(45^\circ + \frac{1}{2} AM)$, because $45^\circ - \frac{1}{2} AM$ is the complement of $45^\circ + \frac{1}{2} AM$, since

$$45^\circ - \tfrac{1}{2} AM = 90^\circ - 45^\circ - \tfrac{1}{2} AM;$$

we have therefore

$$\text{tang}(45^\circ - \tfrac{1}{2} AM) : 1 :: 1 : \text{tang}(45^\circ + \tfrac{1}{2} AM);$$

therefore

$$\text{tang}(45^\circ - \tfrac{1}{2} AM) = \frac{1}{\text{tang}(45^\circ + \frac{1}{2} AM)}.$$

Substituting in the expression for s', we have

$$s' = \log \sqrt{\text{tang}(45^\circ + \tfrac{1}{2} AM)^2} = \log \text{tang}(45^\circ + \tfrac{1}{2} AM)$$
$$= \log. \cot(45^\circ - \tfrac{1}{2} AM).$$

Now $45^\circ - \frac{1}{2} AM =$ half of $90^\circ - AM$, which is the complement of the latitude; we have therefore the increasing latitude

$$s' = \log. \cot(\tfrac{1}{2} \text{ the complement of the latitude}).$$

We therefore take, in the common tables, the logarithm of the cotangent of half the complement of the latitude, and, having multiplied it (112) by 2,30258509, the product will be the increasing latitude expressed in parts of the radius.

But, as it is more conveninent to have the increasing latitude expressed in degrees, it may be thus determined. We found (110), that the length of the semicircumference of a circle, whose radius is 1, is 3,1415926, &c. This number, divided by 180, gives 0,0174533 for the length of a degree. We have therefore only to find how many times this number is contained in the increasing latitude which has just been determined, that is, to divide the increasing latitude by 0,0174533; the quotient will express the increasing latitude in degrees. We obtain therefore the increasing latitude in degrees, by the formula

$$\frac{2{,}30258509 \times \log. \cot(\frac{1}{2} \text{ co. latitude})}{0{,}0174533}.$$

But, by dividing 2,30258509 by 0,0174533, we obtain, as quotient, 131,9283 if we confine ourselves to four decimals. Therefore, *to obtain, in degrees, the increasing latitude, we must take from the common tables the logarithm of the cotangent of half the complement of the latitude, and multiply it by* 131,9283. The product will be the number of degrees and parts of a degree of the increasing latitude.

If, for example, we wish to find the increasing latitude corresponding to 40° of simple latitude, we take half the complement of

40°, that is, half of 50° or 25°. The logarithm of the cotangent of 25° is 0,3313275,† which, being multiplied by 131,9283, gives 43,7115, that is, 43°,7115, or 43° 43′. It is thus we may calculate tables of increasing latitudes.

126. It is by a calculation nearly similar, that we determine the difference of longitude, when we know the difference of latitude and the course. The following is the manner in which it is obtained.

Let OQM (*fig* 52) be the course, Q the point of departure; OA the equator; AMP and $a\,m\,p$, two meridians infinitely near each other. If we imagine the arc $M\,m$ parallel to the equator, the infinitely small triangle $m\,M\,r$ will be rectilineal and right-angled at m; we shall consequently have

$$1 : \text{tang}\, m\,r\,M :: m\,r : M\,m = m\,r\,.\,\text{tang}\, m\,r\,M,$$

supposing the radius $= 1$. If, now, we compare figure 52 with figure 50, we have, agreeably to what we have already seen in the preceding question, $M\,m : A\,a ::$ cos lat : 1, that is,

$$m\,r \times \text{tang}\, m\,r\,M : A\,a :: \text{cos lat} : 1.$$

If we call the sine of the latitude AM, x, its cosine will be $\sqrt{1-x^2}$. The arc $m\,r$, which is the difference of the two latitudes $a\,r$ and AM will be the differential of the arc AM, and will have for its expression $\frac{dx}{\sqrt{1-x^2}}$ (125). If we represent, moreover, by a, the angle $M\,r\,m$, which the course makes with the meridian, or the rhumb, and by z, the difference of longitude BA, which gives $A\,a = d\,z$; the proportion just found is changed into

$$\sqrt{1-x^2} : 1 :: \frac{\text{tang}\, a\,.\,d\,x}{\sqrt{1-x^2}} : d\,z = \text{tang}\, a \times \frac{d\,x}{1-x^2}.$$

But we have just seen, in the preceding question, that the integral of $\frac{dx}{1-x^2}$, was $\frac{1}{2}\, l\, \frac{1+x}{1-x}$; we shall therefore have

$$z = \tfrac{1}{2}\, \text{tang}\, a\,.\, l\, \frac{1+x}{1-x} + C.$$

To determine the constant C, we observe that when $z = 0$, that is, at the point of departure, the latitude AM becomes the latitude

† As we have supposed the radius equal to unity, the logarithms of tangents must be diminished by the logarithm of the radius, which is 10,000000; that is, we must take away the characteristic 10.

of departure BQ. Let t represent the sine of this latitude. The constant C, then, must be such, that, substituting t for x, we may have $z = 0$. We have therefore

$$0 = \tfrac{1}{2} \text{ tang } a \,.\, l\,\frac{1+t}{1-t} + C,$$

and consequently

$$C = -\tfrac{1}{2} \text{ tang } a \,.\, l\,\frac{1+t}{1-t}.$$

Wherefore

$$\begin{aligned} z &= \tfrac{1}{2} \text{ tang } a \,.\, l\,\frac{1+x}{1-x} - \tfrac{1}{2} \text{ tang } a \,.\, l\,\frac{1+t}{1-t} \\ &= \text{tang } a \left(\tfrac{1}{2}\, l\,\frac{1+x}{1-x} - \tfrac{1}{2}\, l\,\frac{1+t}{1-t}\right) \\ &= \text{tang } a \left(l \sqrt{\frac{1+x}{1-x}} - l \sqrt{\frac{1+t}{1-t}} \right). \end{aligned}$$

By reasoning precisely as in the preceding question, we shall find $\sqrt{\frac{1+x}{1-x}}$ is reduced to cot ($\frac{1}{2}$ complement of AM); and for the same reason $\sqrt{\frac{1+t}{1-t}}$ is reduced to cot ($\frac{1}{2}$ complement of BQ), wherefore, *the difference of longitude or* z = *tang* a . (*log. cot* ($\frac{1}{2}$ *co. lat. of point arrived at*) — *log. cot* ($\frac{1}{2}$ *co. lat. of point of departure*); which furnishes a very simple rule for finding the difference of longitude, either by the tables of increasing latitudes, or by the reduced maps, whatever may be the course.

By the Table of Increasing Latitudes.

Find the increasing latitudes corresponding to the latitude of the point arrived at, and of the point of departure. Take the difference of the latitudes thus found, (or their sum, if the latitudes are of different denominations), and multiply it by the tangent of the course; you will have the difference of longitude in degrees, minutes, and seconds.

By the Reduced Maps.

Find upon the meridian the latitude of the point arrived at, and that of the point of departure. Draw, through the extremity of each, a perpendicular meeting the course proposed. The difference of these two perpendiculars, applied to the scale of longitude, will give you, in degrees, minutes, &c. the difference of longitude.

Indeed, AQ, (*fig.* 53) being the meridian; OL the course; AQ, AP the two increasing latitudes; PQ or RT is their difference; now, in the right-angled triangle SRT, we have

$$1 : \text{tang } TRS : : RT : TS;$$

therefore

$$TS = RT \times \text{tang } TRS;$$

now RT is the difference of the two increasing latitudes, and the angle TRS is equal to the angle made by the course with the meridian.

The curve line QM (*fig.* 52), which marks the course of the vessel on the surface of the globe, is called the *Loxodromic.*

In whatever part of the course we suppose the point M, the triangle Mrm has always the same angles, since the angle mrM is always the same, and the angle m is a right angle. There is therefore always the same ratio between Mr and mr, as between the radius and cosine of the angle Mrm, which we have called a. Therefore the number of leagues in the distance QM, is to the number of leagues made in latitude, that is, the number of leagues in IM, as radius is to the cosine of the course. This serves to determine the difference of latitude, when we know the course and distance in leagues. The same proportion serves to determine the course, when we know the difference of latitude and the distance in leagues.

On the manner of reducing, when it is possible, the integration of a proposed differential, to that of a known differential, and distinguishing in what cases this may be done.

128. We shall explain only the method for binomial differentials; it will afterwards be easy to apply the principles to more complex differentials.

Let us suppose, at first, that the proposed differential is

$$h x^s \, dx \, (a + b x^n)^p,$$

and that $x^m dx \, (a + b x^n)^p$ is that on which it is to depend, (or that, to the form of which we would reduce it); that is, let us suppose that the two exponents of the binomial are the same.

We shall suppose

$$\int h x^s \, dx \, (a + b x^n)^p = (a + b x^n)^{p+1} (A x^k + B x^{k+q} + C x^{k+2q}$$
$$+ \ldots\ldots\ldots P x^{k+tq}) + \int R x^m dx \, (a + b x^n)^p;$$

k and q being unknown exponents; t a positive whole number;

A, B, C, P, R, &c. constant coefficients, also unknown. Differentiating, we have

$$h x^s dx (a + b x^n)^p$$
$$= (p+1) n b x^{n-1} dx (a + b x^n)^p (A x^k + B x^{k+q} + C x^{k+2q} \ldots + P x^{k+tq}) + (a + b x^n)^{p+1} (k A x^{k-1} dx + (k+q) B x^{k+q-1} dx + (k+2q) C x^{k+2q-1} dx \ldots\ldots + (k+tq) P x^{k+tq-1} dx) + R x^m dx (a + b x^n)^p,$$

or dividing the whole by $(a + b x^n)^p dx$, we have

$$h x^s = (p+1) n b x^{n-1} (A x^k + B x^{k+q} + C x^{k+2q} \ldots + P x^{k+tq}) + (a + b x^n)(k A x^{k-1} + (k+q) B x^{k+q-1} + (k+2q) C x^{k+2q-1} \ldots\ldots + (k+tq) P x^{k+tq-1}) + R x^m.$$

In order that this equation may be still true, independently of any value of x, it is necessary, that after the multiplications and transpositions are performed, the sum of the quantities which multiply the same power of x be zero; it is by this condition that we determine the coefficients A, B, C, &c. But, that this may take place, the nnmber of powers of x, which enter into this equation, must not exceed the number of these coefficients.

Now the number of coefficients, as may be easily seen, is $t + 2$; let us therefore find the number of the powers of x. In order to that, we must determine k and q.

They may be determined thus. $k - 1$ is the least indeterminate exponent found in the equation; we make it equal to m or to s, according as m or s is the least exponent. The greatest determinate exponent to be found in the equation is, as may be easily seen, $k + tq + n - 1$; we make this equal to s, if we have made

$$k - 1 = m;$$

or to m, if we have made $k - 1 = s$.

Let us suppose $k - 1 = m$, we shall therefore have

$$k + tq + n - 1 = s$$

This done, that the equation may not contain more powers of x, than there are indeterminate coefficients, the coefficients of x in this equation must form an arithmetical progression whose difference is q, which cannot fail to be the case, since we have supposed

$$k - 1 = m, \quad k + tq + n - 1 = s,$$

and t a positive whole number. Now the greatest term of this progression being $k + tq + n - 1$, and the least, $k - 1$, we easily find (*Alg.* 230) the number of terms of this progression to be

$$\frac{k + tq + n - 1 - k + 1}{q} + 1 \text{ or } \frac{tq + n}{q} + 1;$$

therefore
$$\frac{tq + n}{q} + 1 = t + 2$$

and consequently $q = n$; substituting for q and k, their value, in the equation $k + tq + n - 1 = s$, we have $tn + m + n = s$, and consequently

$$t=\frac{s-m-n}{n}=\frac{s-m}{n}-1;$$

wherefore the reduction of one differential to another will be possible, if the difference $s-m$ of the exponents of x out of the two binomials, divided by the exponent of x in the binomial, gives a positive whole number, we then suppose, in the original equation,

$$\int x^s\,dx\,(a+bx^n)^p$$
$$=(a+bx^n)^{p+1}(Ax^{m+1}+Bx^{m+n+1}+Cx^{m+2n+1}$$
$$+\ldots\ldots Px^{s-n+1})+\int Rx^m dx\,(a+bx^n)^p;$$

and, in order to determine the coefficients, A, B, C, P, R, &c. after having differentiated, divided by $(a+bx^n)^p\,dx$, and performed the operations which are indicated, we transpose the whole to one side of the equation, and make the sum of the quantities, which multiply each power of x, equal to zero, which will give as many equations as there are undetermined coefficients.

129. But if we pay attention, we shall find, that when

$$\int x^s\,dx\,(a+bx^n)^p$$

depends on $\int x^m dx\,(a+bx^n)^p$, reciprocally, the latter depends on the former; now, proceeding as above to reduce

$$\int x^m dx\,(a+bx^n)^p \text{ to } \int x^s\,dx\,(a+bx^n)^p,$$

we should find that

$$\frac{m-s}{n}=\text{a positive whole number},$$

and that we must suppose

$$\int x^m dx\,(a+bx^n)^p$$
$$=(a+bx^n)^{p+1}(Ax^{s+1}+Bx^{s+n+1}+\text{\&c.}+Px^{m-n+1})$$
$$+\int Rx^s\,dx\,(a+bx^n)^p,$$

therefore, whether s be greater or less than m, provided that $\frac{s-m}{n}$, or $\frac{m-s}{n}$, gives a positive whole number, we may always reduce one of these differentials to the other by substituting for the first exponent of x in the series Ax^k+Bx^{k+q}, &c. the least of the two exponents m and s, increased by unity, and taking for q the exponent of x in the binomial.

For example, if we wish to reduce

$$x^4\,dx\,(b^2-x^2)^{\frac{1}{2}} \text{ to } dx\,(b^2-x^2)^{\frac{1}{2}},$$

which depends on the quadrature of the circle; we see that $s-m$ is here $4-0$, which, being divided by n, that is, by 2, gives 2, a whole number; the reduction is therefore possible; and since the formula $t=\frac{s-m}{n}-1$ gives $t=1$, and as moreover the least exponent $m=0$, we substitute 1 for k; we then make

$$\int x^4\,dx\,(b^2-x^2)^{\frac{1}{2}}=(b^2-x^2)^{\frac{1}{2}}(Ax+Bx^3)+\int R\,dx\,(b^2-x^2)^{\frac{1}{2}},$$

differentiating, dividing by $(b^2-x^2)^{\frac{1}{2}}dx$, and transposing, we have

$$\begin{aligned} 0=A\,b^2 &- \quad A\,x^2 \quad -3\,B\,x^4 \\ +\ R\ &+3\,B\,b^2x^2-x^4 \\ &-\ 3\,A\,x^2\ -3\,B\,x^4, \end{aligned}$$

whence we deduce

$$-6B-1=0,\ -4A+3Bb^2=0,\ Ab^2+R=0,$$

wherefore,

$$B=-\tfrac{1}{6},A=-\tfrac{1}{8}b^2,R=\tfrac{1}{8}b^4;$$

therefore

$$\int x^4\,dx\,(b^2-x^2)^{\frac{1}{2}}$$

$$=(b^2-x^2)^{\frac{1}{2}}(-\tfrac{1}{8}b^2x-\tfrac{1}{6}x^3)+\tfrac{1}{8}b^4\int dx\,\sqrt{(b^2-x^2)}+C.$$

It is therefore easy, by this method, to find the differentials which are referred to a given differential, and consequently those which are referred to the quadrature of the circle, of the ellipse, and the hyperbola, differentials of which it is easy to find the different expressions, by means of the different equations of these curves.

130. We may here take occasion to observe, that this method shows also the binomial differentials which are integrable; indeed, to find, among such binomial differentials, as $h\,x^s\,dx\,(a+b\,x^n)^p$, those which are integrable, is to find those which depend on

$$R\,x^{n-1}dx\,(a+b\,x^n)^p,$$

which has been found (90) to be directly integrable; now it results from what is laid down (128) that $\frac{s-n+1}{n}$ must be a positive whole number, that is, that

$$\frac{s+1}{n}=\text{a positive whole number},$$

which agrees with what has been said (91).

131. Let us now suppose that the two binomials which enter into the differentials in question, have different exponents, so that the proposed differential is $h\,x^s\,dx\,(a+b\,x^n)^r$, and that, to whose form we wish to reduce it, is $x^m dx\,(a+b\,x^n)^p$, p having a numerical value less than r. If r is positive, we change the differential

$$h\,x^s\,dx\,(a+b\,x^n)^r \text{ into } h\,x^s\,dx\,(a+b\,x^n)^{r-p}\times(a+b\,x^n)^p.$$

Then if $r-p$ is a positive whole number, we may reduce

$$h\,x^s\,dx\,(a+b\,x^n)^{r-p}(a+b\,x^n)^p$$

to a series of terms of this form

$$(A'\,x^s+B'\,x^{s+n}+C'\,x^{s+2r}+\&c.)\,dx\,(a+b\,x^n)^p,$$

of which each may be reduced to the form $x^m\,dx\,(a+b\,x^n)^p$ by the preceding method, if $s-m$ can be divided by n, and to reduce the whole to that form, we must follow exactly the method there given, taking for the quantity which was there called s the greatest expo-

nent of x in the expanded value of

$$h\,x^s\,dx\,(a+b\,x^n)^{r-p}.$$

If we had, for example,

$$\int x^2\,dx\,(b^2-x^2)^{\frac{3}{2}} \text{ to reduce to} \int dx\,(b^2-x^2)^{\frac{1}{2}},$$

we should change $\int x^2\,dx\,(b^2-x^2)^{\frac{3}{2}}$ into

$$\int x^2\,dx\,(b^2-x^2)\,(b^2-x^2)^{\frac{1}{2}} \text{ or} \int (b^2\,x^2\,dx-x^4\,dx)\,(b^2-x^2)^{\frac{1}{2}},$$

then what we have to take for s, is 4. We suppose, therefore, conformably to the method,

$$\int (b^2\,x^2\,dx-x^4\,dx)\,(b^2-x^2)^{\frac{1}{2}}$$
$$=(b^2-x^2)^{\frac{3}{2}}\,(A\,x+B\,x^3)+\int R\,dx\,(b^2-x^2)^{\frac{1}{2}}$$

If, on the contrary, the value of r is negative, the differential, to which the proposed differential is to be referred, must be prepared thus, $x^m dx\,(a+b\,x^n)^{p-r}\times(a+b\,x^n)^p$, then, if $p-r$ is a whole number, as it will necessarily be positive (since we suppose r negative and greater than p, whatever be the value of p) we may reduce $x^m dx\,(a+b\,x^n)^{p-r}(a+b\,x^n)^r$ to a finite series of terms of this form, $(A'x+B'x^{m+n}+C'x^{m+2n}+\text{\&c.})\,(a+b\,x^n)^r$, we may then proceed as if it were required to reduce this last to the form $x^s\,dx\,(a+b\,x^n)^r$, that is, we may proceed in a manner precisely similar to that to be followed in the case when r is positive.

If it were required, for example, to reduce

$$g\,x^{-2}\,dx\,(a^2+x^2)^{-2} \text{ to } dx\,(a^2+x^2)^{-1} \text{ or } \frac{dx}{a^2+x^2},$$

which (110) is integrated by means of an arc of a circle, of which x is the tangent, and a the radius, we should change $dx\,(a^2+x^2)^{-1}$ into $(a^2+x^2)\,dx\,(a^2+x^2)^{-2}$, and, as the least exponent out of the proposed binomial is 2, we should suppose

$$\int R\,(a^2+x^2)\,dx\,(a^2+x^2)^{-2}$$
$$=(a^2+x^2)^{-1}\,(A\,x^{-1}+B\,x)+\int g\,x^{-2}\,dx\,(a^2+x^2)^{-2}.$$

And we should proceed as above to determine the coefficients, A, B, and R. Then, by transposition, we should have the value of

$$\int g\,x^{-2}\,dx\,(a^2+x^2)^{-2},$$

in which we should afterwards reduce

$$R\,(a^2+x^2)\,dx\,(a^2+x^2)^{-2} \text{ to } R\,dx\,(a^2+x^2)^{-1}.$$

On Rational Fractions.

132. Every rational differential fraction is always integrable either algebraically, or by arcs of a circle, or logarithms, or all these means united, or by only two of them.

They are always integrable algebraically, when they have no varible denominator unless it be a simple quantity, excepting only the case

in which the denominator is raised no higher than to the first power, as we have already seen (82).

It remains for us therefore to show the truth of our assertion, in the other cases; that is, when the proposed differential has a rational complex denominator.

We shall suppose that the variable in the numerator of the proposed differential fraction is of a lower degree than in the denominator. If this were not the case, we should make it so, by dividing the numerator by the denominator until the remaining power should be less than in the denominator. For example, if we had to integrate $\frac{x^3\,dx}{a^2+3\,a\,x+x^2}$, we should begin by dividing $x^3\,dx$ by $a^2+3\,a\,x+x^2$; we should find $x\,dx$ for a quotient and $-3\,a\,x^2\,dx - a^2\,x\,dx$ for a remainder; we should again divide this remainder by the same denominator, and find $-3\,a\,dx$ for a quotient, and

$$+8\,a^2\,x\,dx + 3\,a^3\,dx$$

for a remainder; then instead of $\frac{x^3\,dx}{a^2+3\,a\,x+x^2}$, we should take

$$x\,dx - 3\,a\,dx + \frac{8\,a^2\,x\,dx + 3\,a^3\,dx}{a^2+3\,a\,x+x^2}.$$

In order to discover by what means we may integrate rational differential fractions, we recollect that the differential of the logarithm of a quantity, being the differential of that quantity divided by the quantity itself, that is, being always a fraction, it is very natural to suspect that the integration of rational fractions often depends on logarithms. Let us take, for example,

$$2\,a\,l\,(a+x) - 2\,a\,l\,(2\,a+x);$$

differentiating, we have

$$\frac{2\,a\,dx}{a+x} - \frac{2\,a\,dx}{2\,a+x},$$

or, reducing to the same denominator,

$$\frac{2\,a^2\,dx}{2\,a^2+3\,a\,x+x^2}.$$

Now, it is evident that, in order to integrate this fraction, we have only to decompose it into two fractions, one of which shall have $a+x$, and the other $2\,a+x$ for a denominator. The numerators will be constant numbers multiplied by dx; these two fractions would then be integrated by logarithms.

133. It is therefore very natural to endeavour, in order to integrate fractions of this kind, to decompose them into as many simple fractions as the denominator has factors, each one of which shall have for its denominator one of these factors. This, indeed, is the method which we may and must pursue, when all the factors of which the denominator may have been formed are unequal.

134. But when, among the factors of the denominator, there are any which are equal to each other, then we are not to expect that the method should be successful, because the integral cannot en-

tirely depend on logarithms. If, for example, we had, $\frac{dx}{(a+x)^2}$, whose denominator has two equal factors $a+x$ and $a+x$, we should find (88) that the integral of this quantity or of its equal

$$dx\,(a+x)^{-2} \text{ is} -(a+x)^{-1}+C,$$

which does not depend on logarithms. But we see, at the same time, that if we should differentiate such a quantity as

$$\frac{a^2}{a+x}+2al(a+x)+2al(2a+x)-al(3a+x),$$

we should have

$$\frac{-a^2\,dx}{(a+x)^2}+\frac{2a\,dx}{a+x}+\frac{2a\,dx}{2a+x}-\frac{a\,dx}{3a+x},$$

or

$$\frac{(2ax+a^2)\,dx}{(a+x)^2}+\frac{2a\,dx}{2a+x}-\frac{a\,dx}{3a+x},$$

or, reducing the whole to a common denominator,

$$\frac{10a^4\,dx+26a^3x\,dx+17a^2x^2\,dx+3ax^3\,dx}{(a+x)^2\,(2a+x)\,(3a+x)};$$

a fraction which, in order to be integrated, would only require to be reduced to

$$\frac{2ax+a^2}{(a+x)^2}\,dx+\frac{2a\,dx}{2a+x}-\frac{a\,dx}{3a+x},$$

that is, to be decomposed into three fractions, of which the first should have for its denominator all the equal factors, and in its numerator all the powers of x less than the highest power of the denominator. The other two fractions should have each, for its denominator, one of the unequal factors, and no power of x in its numerator. In this manner every rational fraction may be integrated; and we proceed in this manner, at least when there are no imaginary factors in the denominator; which case will be examined hereafter.

Thus $\frac{(a+bx+cx^2+\ldots\,kx^{n-1})\,dx}{(M+Nx+Bx^2+\ldots\,Tx^n)}$, representing, in general, any rational fraction; if we suppose the denominator to have a number m of factors, equal to $x+g$, a number p of factors, equal to $x+h$, &c. and any number of unequal factors, represented by $x+i$, $x+q$, $x+r$, &c. the proposed fraction will be

$$\frac{(a+bx+cx^2+\ldots\,kx^{n-1})\,dx}{(x+g)^m(x+h)^p\times\text{\&c.}\,(x+i)\,(x+q)\,(x+r)\,\text{\&c.}}.$$

In order to integrate this fraction, we must suppose it equal to

$$\frac{Ax^{m-1}dx+Bx^{m-2}dx\ldots+R\,dx}{(x+g)^m}$$

$$+\frac{A'x^{p-1}dx+B'x^{p-2}dx+\ldots R'dx}{(x+h)^p},\ \text{\&c.}$$

$$+\frac{L\,dx}{x+i}+\frac{M\,dx}{x+q}+\frac{N\,dx}{x+r}+\text{\&c.}$$

A, B, C, &c. being constant and undetermined coefficients. If, then, we can by any means determine these coefficients, it will be easy to find the integral. This is evident in the case of the simple fractions $\frac{L\,dx}{x+i}$, $\frac{M\,dx}{x+q}$, $\frac{N\,dx}{x+r}$, &c. of which the integral is

$L\,l\,(x+i)$, $M\,l(x+q)$, $N\,l\,(x+r)$, &c. As to the fraction

$$\frac{A\,x^{m-1}\,dx+B\,x^{m-2}\,dx+\ldots R\,dx}{(x+g)^m};$$

we take, for the sake of greater simplicity, $x+g=z$, which gives $x=z-g$, and $dx=dz$. By substituting these values, we reduce the whole to a series of simple quantities easily integrated, one only of which will have the form $\frac{dz}{z}$, that is, be integrated by logarithms.

In like manner, for the terms

$$\frac{A'\,x^{p-1}\,dx+B'\,x^{p-2}\,dx\ldots\ldots R'\,dx}{(x+h)^p},\text{ we take } x+h=z'.$$

There thus remain only two things to examine; the first is how to find the factors of the denominator of the proposed differential fraction; the second, how to find the undetermined coefficients.

135. To find the factors of the denominator, we proceed as we should to resolve the equation produced by setting the denominator equal to zero, since (*Alg.* 184) to resolve an equation is to find the binomial factors, by the multiplication of which the equation was formed. Thus we must employ the methods given (*Alg.* 185, &c.)

136. As to the manner of finding the coefficients A, B, C, the way which offers itself as most natural is to reduce all the fractions in which they occur to the same denominator, then both members of the equation formed of the proposed fraction and these new fractions, having the same denominator, we may suppress this denominator, and having transposed the whole to one side of the equation, we shall find that, in order that the equation should be true, independently of any value of x, the sum of the factors, which multiply each power of x, must be equal to zero. This condition will give as many equations as there are undetermined coefficients, and by means of them the coefficients may be determined. The following are some examples.

Let it be proposed to integrate $\frac{dx}{a^2-x^2}$; we suppose $\frac{dx}{a^2-x^2}$ $=\frac{A\,dx}{a+x}+\frac{B\,dx}{a-x}$, since the two factors of the denominator are $a+x$ and $a-x$, then reducing to the same denominator, we have $\frac{dx}{a^2-x^2}=\frac{(A\,a-A\,x+B\,a+B\,x)\,dx}{a^2-x^2}$, suppressing the common denominator, dividing by dx, and transposing, we have

$$\left.\begin{array}{r}1+A\,x\\ -A\,a-B\,x\\ -B\,a\end{array}\right\}=0;$$

wherefore $1-A\,a-B\,a=0$, and $A\,x-B\,x=0$; from the last of these we have $A=B$, wherefore, the first becomes

$$1 - A\,a - A\,a = 0, \text{ or } 1 - 2\,A\,a = 0,$$

whence $A = \frac{1}{2\,a}$, and $B = \frac{1}{2\,a}$; we have therefore

$$\frac{d\,x}{a^2 - x^2} = \frac{\frac{1}{2\,a}d\,x}{a + x} + \frac{\frac{1}{2\,a}d\,x}{a - x},$$

of which the integral is

$$\int \frac{d\,x}{a^2 + x^2} = \frac{1}{2\,a}\,l\,(a + x) - \frac{1}{2\,a}\,l\,(a - x) + C = \frac{1}{2\,a}\;l\,\frac{a + x}{a - x} + C.$$

Let us take, as a second example, the fraction

$$\frac{10\,a^4\,d\,x + 26\,a^3\,x\,d\,x + 17\,a^2\,x^2\,d\,x + 3\,a\,x^3\,d\,x}{(a + x)^2\,(2\,a + x)\,(3\,a + x)},$$

which was found (134) by differentiating $\frac{a^2}{a + x} + 2\,a\,l\,(a + x)$ $+ 2\,a\,l\,(2\,a + x) - a\,l\,(3\,a + x)$, we shall then suppose

$$\frac{10\,a^4\,d\,x + 26\,a^3\,x\,d\,x + 17\,a^2\,x^2\,d\,x + 3\,a\,x^3\,d\,x}{(a + x)^2\;(2\,a + x)\;(3\,a + x)}$$
$$= \frac{(A\,x + B)\,d\,x}{(a + x)^2} + \frac{C\,d\,x}{2\,a + x} + \frac{D\,d\,x}{3\,a + x},$$

reducing to the same denominator, suppressing the common denominator, dividing by $d\,x$, and transposing, we have

$$\left.\begin{array}{rrrr} 10\,a^4 & + 26\,a^3\,x & + 17\,a^2\,x^2 & + 3\,a\,x^3 \\ - 6\,B\,a^2 & - 5\,B\,a\,x & - B\,x^2 & - A\,x^3 \\ - 3\,C\,a^3 & - 6\,A\,a^2\,x & - 5\,A\,a\,x^2 & - C\,x^3 \\ - 2\,D\,a^3 & - 7\,C\,a^2\,x & - 5\,C\,a\,x^2 & - D\,x^3 \\ & - 5\,D\,a^2\,x & - 4\,D\,a\,x^2 & \end{array}\right\} = 0$$

therefore

$$3\,a - A - C - D = 0,\; 17\,a^2 - B - 5\,A\,a - 5\,C\,a - 4\,D\,a = 0,$$
$$26\,a^3 - 5\,B\,a - 6\,A\,a^2 - 7\,C\,a^2 - 5\,D\,a^2 = 0,$$
$$10\,a^4 - 6\,B\,a^2 - 3\,C\,a^3 - 2\,D\,a^3 = 0,$$

equations from which we deduce †

$$A = 2\,a,\; B = a^2,\; C = 2\,a,\; D = -a;$$

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† These values are found in the following manner.

$$A = 3\,a - C - D;$$
$$B = 17\,a^2 - 15\,a^2 + 5\,C\,a + 5\,D\,a - 5\,C\,a - 4\,D\,a$$
$$B = 2\,a^2 + D\,a$$
$$C = \frac{26\,a^3 - 5\,B\,a - 6\,A\,a^2 - 5\,D\,a^2}{7\,a^2}$$
$$C = \frac{26\,a^3 - 10\,a^3 - 5\,D\,a^2 - 18\,a^3 + 6\,C\,a^2 + 6\,D\,a^2 - 5\,D\,a^2}{7\,a^2}$$
$$7\,C = -2\,a + 6\,C - 4\,D$$
$$C = -2\,a - 4\,D$$
$$D = \frac{10\,a^4 - 6\,B\,a^2 - 3\,C\,a^3}{2\,a^3}$$

the proposed differential is therefore changed into

$$\frac{(2ax+a^2)\,dx}{(a+x)^2}+\frac{2a\,dx}{2a+x}-\frac{a\,dx}{3a+x},$$

precisely as we found it above. The two last terms have evidently for their integral $2a\,l\,(2a+x)-a\,l\,(3a+x)$; with regard to the term $\frac{2ax+a^2}{(a+x)^2}dx$, we make $a+x=z$, and have $x=z-a$, and $dx=dz$; whence we have

$$\frac{(2az-a^2)\,dz}{z^2}\text{ or }\frac{2a\,dz}{z}-\frac{a^2\,dz}{z^2},$$

of which the integral is $2a\,lz+\frac{a^2}{z}$ or $2a\,l\,(a+x)+\frac{a^2}{a+x}$; the whole integral is $\frac{a^2}{a+x}+2a\,l(a+x)+2a\,l\,(2a+x)-a\,l\,(3a+x)$, as it should be.

137. This method is general. But there are several shorter ways of finding the coefficients. We may, for example, find the coefficients of simple fractions, independently of each other, in the following manner. Let $\frac{N\,dx}{M}$ be the fraction proposed; $hx+a$, one of the factors of the denominator; let P represent the other factors, or be the quotient of M divided by $hx-a$. Conceive $\frac{N\,dx}{M}$ to be decomposed into $\frac{A\,dx}{hx+a}+\frac{Q\,dx}{P}$; we shall have

$$\frac{N\,dx}{M}=\frac{A\,dx}{hx+a}+\frac{Q\,dx}{P},\text{ or }\frac{N}{M}=\frac{A}{hx+a}+\frac{Q}{P};$$

therefore, by reducing to a common denominator, observing that

$$P=\frac{M}{hx+a}\text{ or }P\times(hx+a)=M,$$

we shall have $N=AP+Q\,(hx+a)$. But if we differentiate the equation $(hx+a)\,P=M$, we have $h\,P\,dx+(hx+a)\,dP=dM$. Now as this equation and the equation $N=AP+Q\,(hx+a)$ must be true for every value of x, they must be true, if we give to x any value whatever. We therefore give to x the value which gives the most simple result, that is, the value $-\frac{a}{h}$ obtained by supposing the

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$$=\frac{10a^4-12a^4-6Da^3+6a^4+12Da^3}{2a^3}$$

$$=5a-6a-3D+3a+6D\;\therefore\;-2a=2D$$

$$D=-a$$

$C=-2a-4D=-2a+4a=2a$
$B=2a^2+Da=2a^2-a^2=a^2$
$A=3a-C-D=3a-2a+a=2a;$

denominator $hx+a=0$. We then have $hP\,dx=dM$, and $N=AP$. Substituting in the second the value $P=\frac{dM}{h\,dx}$, given by the first, and we have $A=\frac{hN\,dx}{dM}$; that is, to obtain the numerator A of any one of the simple fractions, we must divide the numerator $N\,dx$ of the proposed fraction by dM the differential of its denominator, and having substituted for x the value obtained by making the denominator of the simple fraction equal to zéro, multiply the whole by the coefficient of x in this denominator.

To obtain, for example, the value of the numerators A and B of the fractions $\frac{A\,dx}{a+x}$ and $\frac{B\,dx}{a-x}$, into which we above resolved the fraction $\frac{dx}{a^2-x^2}$, we differentiate the denominator a^2-x^2, which gives $-2x\,dx$. We then divide the numerator dx of the proposed fraction by $-2x\,dx$, which gives $-\frac{1}{2x}$, in which successively substituting for x, $-a$ and a, (which are the values obtained by making the denominators $a+x$ and $a-x$ of the partial fractions successively equal to zero), and multiplying by 1 and -1, the values of h, we have $\frac{1}{2a}$, and $\frac{1}{2a}$ for the values of A and B, as was found above.

We might also find general rules for ascertaining the coefficients of the numerators of the partial fractions which have for their denominator the product of the equal roots; but shall not now stop to investigate them.

138. Although the rules just given for integrating rational fractions be general, yet when some of the factors of the denominator are imaginary, we have, for an integral, quantities composed of imaginary ones. Such an integral is not the less real, though it is sometimes with difficulty reduced to a real form. In this case, we first take out all the real factors of the denominator, and then decompose the remainder into factors not of the first but of the second degree, which are always real. Then, for each factor of the second degree, which may always be represented by ax^2+bx+c, we form a fraction of this form $\frac{Ax\,dx+B\,dx}{ax^2+bx+c}$, and determine the coefficients as above.

139. If among the factors of the second degree, there are found any which are equal to each other, we form, for each group of these equal factors, a fraction of the form

$$\frac{Ax^{2n-1}dx+Bx^{2n-2}dx+\ldots.Q\,dx}{(ax^2+bx+c)^n},$$

n being the number of equal factors ax^2+bx+c.

140. It only remains to show how these quantities may be integrated.

With regard to the first, let us suppose, for the sake of making the operations more simple, that the partial fraction is reduced to the

form $\frac{A' x\,dx + B' dx}{x^2 + a' x + b'}$, which may be always done by dividing both terms by a.

We then cause the second term of the denominator to disappear by making $x + \frac{1}{2} a' = z$; which gives $x = z - \frac{1}{2} a'$, and $dx = dz$; by substituting these values we obtain a quantity of the form

$$\frac{C z\,dz + D\,dz}{z^2 + q^2},$$

of which, the first part $\frac{C z\,dz}{z^2 + q^2}$ is integrated by logarithms (124) and the second by means of an arc of a circle, whose radius is q and tangent z.

As to the quantities which have the form

$$\frac{A x^{2n-1} dx + B x^{2n-2} dx + \ldots . Q\,dx}{(x^2 + 2ax + b)^n},$$

we make the second term of the denominator disappear, and obtain a quantity of the form

$$\frac{M z^{2n-1} dz + N z^{2n-2} dz + \ldots T\,dz}{(z^2 + q^2)^n},$$

which is integrated by reducing to the form $\frac{dz}{z^2 + q^2}$, by the method given (131), the integral of the sum of the terms, in which z has even exponents. Those whose exponent is odd may be integrated by what is given in article 91, or by being reduced to $\frac{z\,dz}{(z^2 + q^2)^n}$, according to the method given (130).

Thus every rational fraction is either integrated exactly, or depends only on arcs of a circle or logarithms.

On certain Transformations, by which the integration may be facilitated.

141. On this subject no general rules can be given. The inspection of the quantities, experience, and practical address will dictate, on each occasion, what is best to be done.

The object of the transformations here spoken of is to render the proposed differentials rational, as we then know how to integrate them. We subjoin, however, a few observations.

142. If there are no radical quantities but such as are simple quantities, we first give them fractional exponents, which we reduce to the same denominator. Then, if $x^{\frac{k}{l}}$ represent one of these quantities so prepared, we make $x^{\frac{1}{l}} = z$, which gives $x = z^l$, and $dx = l z^{l-1} dz$. We substitute these values, and obtain a quantity entirely rational. If we have, for example,

$$\frac{dx\sqrt{x} + a\,dx}{\sqrt[3]{x^2} + \sqrt{x}},$$

we give it the form $\dfrac{x^{\frac{1}{2}}dx+a\,dx}{x^{\frac{2}{3}}+x^{\frac{1}{2}}}$, which we change into

$$\frac{x^{\frac{3}{6}}dx+a\,dx}{x^{\frac{4}{6}}+x^{\frac{3}{6}}}.$$

Then making $x^{\frac{1}{6}}=z$, we have $x=z^6$, $dx=6z^5\,dz$, and consequently

$$\frac{6z^8\,dz+6az^5\,dz}{z^4+z^3},$$

which is reduced to

$$\frac{6z^5\,dz+6az^2\,dz}{z+1},$$

and easily integrated by the rules already given for rational fractions.

143. Every quantity, in which there is only a complex radical not exceeding the second degree, and in which the variable under the radical sign does not exceed the second degree, may always be rendered rational by one or other of the two following methods: 1. After having freed from the radical sign the square of the variable under the radical sign, we make this radical equal to the same variable plus or minus another variable. 2. We decompose the quantity affected by the radical sign into its two factors, and make it, after being reduced to this form, equal to one of its factors multiplied by a new variable.

If we had, for example, $\dfrac{dx}{\sqrt{x^2-a^2}}$, we might make $\sqrt{x^2-a^2}=x-z$;

then $x=\dfrac{z^2+a^2}{2z}$. Whence $dx=\dfrac{(z^2-a^2)\,dz}{2z^2}$, and

$$\sqrt{x^2-a^2}=\frac{a^2-z^2}{2z}=-\frac{(z^2-a^2)}{2z};$$

whence $\dfrac{dx}{\sqrt{x^2-a^2}}=-\dfrac{dz}{z}$, which is easily integrated.

We might also, in this same example, make

$\sqrt{x^2-a^2}$, or $\sqrt{(x-a)(x+a)}=(x-a)\,z$; then, squaring, and dividing by $x-a$, $x+a=(x-a)\,z^2$; whence $x=\dfrac{a+az^2}{z^2-1}$;

$\sqrt{x^2-a^2}=\dfrac{2az}{z^2-1}$; $dx=\dfrac{-4az\,dz}{(z^2-1)^2}$; therefore

$$\frac{dx}{\sqrt{x^2-a^2}}=\frac{-2\,dz}{z^2-1},$$

which is integrated by the rules given above for rational fractions.

These methods may be applied to the rectification of the parabola, of which the element $\sqrt{dx^2+dy^2}$ is $\sqrt{dy^2+\dfrac{4y^2\,dy^2}{p^2}}$ or

$$dy\sqrt{1+\frac{4y^2}{p^2}}.$$

We first free y^2 of its factors by writing it $\frac{2dy}{p}\sqrt{\frac{p^2}{4}+y^2}$; and then make $\sqrt{\frac{p^2}{4}+y^2}=y+z$.

144. When there is no radical but a square root, and no powers of x but even powers, we make the radical equal to a new variable multiplied by the given variable. If, for example, we had $\frac{dx}{\sqrt{a^2-x^2}}$, we might make $\sqrt{a^2-x^2}=xz$. If there were a second term under the radical sign, we might, notwithstanding, make use of this transformation, having first made the second term disappear, at least when there is no power of x without the radical sign.

145. Finally, we may, with a view to making a quantity rational, put the variable or any fraction of the variable equal to a new variable or some fraction of one, in which we leave something indeterminate, which may serve to effect the object in view. For example, to ascertain in what case we might render the quantity

$$x^m dx(a+bx^n)^p$$

rational, we should make $(a+bx^n)^p=z^q$, q being indeterminate.

We should have $a+bx^n=z^{\frac{q}{p}}$; $x^n=\frac{z^{\frac{q}{p}}-a}{b}$;

$$x=\left(\frac{z^{\frac{q}{p}}-a}{b}\right)^{\frac{1}{n}},\quad x^m=\left(\frac{z^{\frac{q}{p}}-a}{b}\right)^{\frac{m}{n}};$$

$$dx=\frac{q}{npb}\,z^{\frac{q}{p}-1}\,dz\left(\frac{z^{\frac{q}{p}}-a}{b}\right)^{\frac{1}{n}-1};$$

therefore

$$x^m\,dx(a+bx^n)^p=\frac{q}{npb}\,z^{\frac{q}{p}+q-1}\,dz\left(\frac{z^{\frac{q}{p}}-a}{b}\right)^{\frac{m}{n}+\frac{1}{n}-1};$$

which is integrable, whatever may be the value of q, when $\frac{m+1}{n}-1$ is a positive whole number or zero (85), and which may be rendered rational by making $q=p$, when $\frac{m+1}{n}-1$ is a negative whole number. And if the value of p is $\pm\frac{k}{2}$, k being an odd whole number, we may reduce it to the case mentioned (143), by making $q=k$, if $\frac{m+1}{n}$ has for its value $\pm\frac{k'}{2}$, k' being an odd whole number.

146. We shall extend transformations of this kind no farther.

We shall only observe that certain integrations are often facilitated by making the variable equal to a fraction, such as $\frac{1}{z}$. For example, if we had $\frac{x^{15}\,dx + a\,dx}{x^{20}+x^{18}}$; by making $x = \frac{1}{z}$, we should have $\frac{-z^3\,dz - a\,z^{18}\,dz}{1+z^2}$, which may be reduced by division to a series of simple quantities, and a quantity of the form $\frac{A\,dz}{1+z^2}$, of which we already know the integral.

On the Integration of Exponential Quantities.

147. There are no other rules to be given on the differentiation of these quantities, than that we should endeavour to decompose them into two factors, one of which shall be the differential of the logarithm of the other, or a constant part of it (27); and then divide by the differential of the logarithm of this second factor. Thus we see that $x^y\left(dy\,lx + \frac{y\,dx}{x}\right)$ is integrable, because the factor $dy\,lx + \frac{y\,dx}{x}$ is the differential of $y\,l\,x$, the logarithm of x^y; we therefore have, as the integral, $x^y\,\frac{\left(dy\,lx + \frac{y\,dx}{x}\right)}{d\,(l\,x^y)} + C$; that is,

$$x^y\,\frac{\left(dy\,lx + \frac{y\,dx}{x}\right)}{dy\,lx + \frac{y\,dx}{x}} + C,$$

or $x^y + C$. By the same rule, we see that $dx\,e^{ax}$ is integrable, because dx is the differential of the logarithm of e^{ax}, divided by a constant quantity. We have therefore

$$\int dx\,e^{ax} = \frac{dx\,e^{ax}}{a\,dx\,l\,e} = \frac{e^{ax}}{a\,l\,e}.$$

When e is the number whose logarithm is 1, the rule is reduced to dividing the proposed differential by the differential of the exponent of e.

If we had $x^m dx\,e^{ax}$ to integrate, e being the number whose logarithm is 1, we might do it, when m is a positive whole number, by making

$$\int x^m dx\,e^{ax} = e^{ax}\,(A\,x^m + B\,x^{m-1} + E\,x^{m-2} + \&c. + k).$$

If, for example, we have $x^2\,dx\,e^{ax}$, we suppose

$$\int x^2\,dx\,e^{ax} = e^{ax}\,(A\,x^2 + B\,x + E).$$

Differentiating (27), and then dividing by $dx\,e^{ax}$, we have

$$x^2 = Aax^2 + aBx + aE \\ + 2Ax + B;$$

whence $Aa = 1$, $aB + 2A = 0$, $aE + B = 0$; that is,

$$A = \frac{1}{a},\ B = \frac{-2}{a^2},\ E = \frac{2}{a^3},$$

wherefore the integral $\int x^2\,dx\,e^{ax}$ is

$$e^{ax}\left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right) + C.$$

The number e, whose logarithm is 1, may be employed with advantage for the integration of many quantities, especially when they contain logarithms. For example, if we had to integrate $x^n\,dx\,(lx)^m$, we should make $lx = z = z\,le$; wherefore $x = e^z$; $dx = dz\,e^z$; and consequently $x^n\,dx\,(lx)^m = z^m dz\,e^{(n+1)z}$, which is integrated in the same case as the preceding and in the same manner.

On the Integration of Quantities with two or more Variables.

148. If we examine the rule given for differentiating quantities with several variables, we shall see, that in order to integrate differentials with several variables (when it is possible), we must collect all the terms affected by the differential of a single variable, and integrate them as if there were no other variable but that, that is, as if all the others were constant If we then differentiate this integral, making all the variables vary successively, and subtract the result from the proposed differential, the integral, thus found, is, after a constant quantity is added, the true integral, provided there be no remainder. If there be a remainder, it will not contain the variable, with reference to which the integration has been performed; we pursue with the remainder the same process as before, and so on with each variable. If, for example, we had

$$3x^2 y\,dx + x^3\,dy + 5xy^4\,dy + y^5\,dx;$$

we should take the two terms affected by dx, viz.

$$3x^2\,y\,dx + y^5\,dx,$$

and integrate them as if y were constant. The integral is

$$x^3\,y + y^5\,x.$$

Now this quantity, being differentiated with reference to x and y, and the result subtracted from the proposed differential, nothing remains; we thence conclude that the integral is $x^3\,y + y^5\,x + C$.

If we had

$$x^3\,dy + 3x^2\,y\,dx + x^2\,dz + 2xz\,dx + x\,dx + y^2\,dy;$$

by collecting all the terms affected by dx, and integrating, x and y being regarded as constant, we should have $x^3\,y + x^2\,z + \frac{x^2}{2}$. Subtracting the differential of this quantity, considered as having three variables, from the proposed differential, and there remains $y^2\,dy$; we therefore take the integral of $y^2\,d\,y$, which is $\frac{y^3}{3}$, and adding it, together with a constant, to the integral already found, we have

$$x^3 y + x^2 z + \frac{x^2}{2} + \frac{y^3}{3} + C,$$

for the integral.

149. But as it is not always possible to integrate a differential with several variables, it will be well to point out a character, by which it may be known when it is so.

150. To this end, it must be observed, that if, in any quantity Q, composed in any manner of two other quantities x and y, we at first substitute for x, a certain quantity p, and in the result substitute q for y, we shall have the same final result, as if we had first substituted q for y, and afterwards p for x. This is evident.

151. It thence follows, that if we differentiate a quantity Q composed of x, y, and constants, making first only x variable, and then differentiate the result, making only y variable, we shall have the same final result, as if we had first differentiated, making only y variable, and afterwards differentiated this result, making only x variable.

Indeed, let us conceive, that by substituting first $x + dx$ for x, Q becomes Q'; we have $Q' - Q$ for the differential. If, by substituting $y + dy$ for y in this quantity, Q' becomes Q'', and Q becomes Q''', so that $Q' - Q$ becomes $Q'' - Q'''$, we shall have $Q'' - Q''' - Q' + Q$ for the second differential.

Suppose now we substitute in the contrary order, and since, by substituting $y + dy$, instead of y, in Q, it becomes Q''', we shall have $Q''' - Q$ for the first differential, on the supposition that y is variable. If we now substitute $x + dx$, instead of x, in this quanty, Q will become Q', as above, and Q''' will become Q'' (150), so that $Q''' - Q$ will become $Q'' - Q'$, wherefore the second differential will be $Q'' - Q' - Q''' + Q$, precisely the same as before.

Let us now suppose, that A representing a quantity composed of x and y, $\frac{dA}{dy} dy$ indicates the differential of A, taken by making y only variable; and $\frac{dA}{dx} dx$ that of A, making x alone variable. In like manner, $\frac{ddA}{dx\,dy} dx\,dy$ will indicate that A was first differentiated, supposing only x variable, and the result was then differentiated, supposing only y variable.

152. These explanations being thus given, let $A\,dx + B\,dy$ be an exact differential, and M its integral; we shall therefore have

$$\frac{dM\,dx}{dx} + \frac{dM\,dy}{dy} = A\,dx + B\,dy;$$

therefore

$$\frac{dM}{dx} = A, \text{ and } \frac{dM}{dy} = B;$$

therefore also

$$\frac{ddM}{dx\,dy} dy = \frac{dA}{dy} dy; \text{ and } \frac{ddM}{dy\,dx} dx = \frac{dB}{dx} dx;$$

or $$\frac{ddM}{dx\,dy}=\frac{dA}{dy}, \text{ and } \frac{ddM}{dy\,dx}=\frac{dB}{dx};$$

but it has just been demonstrated (151) that

$$\frac{ddM\,dx\,dy}{dx\,dy}=\frac{ddM\,dy\,dx}{dy\,dx};$$

wherefore $\frac{ddM}{dx\,dy}=\frac{ddM}{dy\,dx}$; wherefore also $\frac{dA}{dy}=\frac{dB}{dx}$, that is, if $A\,dx+B\,dy$ be a complete differential, the differential of A found by making only y vary and dividing by dy, must be equal to the differential of B found by making only x vary and dividing by dx.

Thus we perceive that $\frac{1}{3}y^3\,dx+xy^2\,dy$ is a complete differential, because $\frac{d(\frac{1}{3}y^3)}{dy}=\frac{d(xy^2)}{dx}$; in fact, the first member is reduced to $\frac{y^2\,dy}{dy}$, and the second to $\frac{y^2\,dx}{dx}$. We perceive, on the contrary, that $xy\,dx+2x\,dy$ is not integrable because $\frac{d(xy)}{dy}$ is not equal to $\frac{d(2x)}{dx}$.

153. If more than two variables enter into the proposed differential, that is, if it be of the form

$$A\,dx+B\,dy+C\,dz,$$

it is necessary, in order that it be integrable, that we have

$$\frac{dA}{dy}=\frac{dB}{dx},\ \frac{dA}{dz}=\frac{dC}{dx},\ \frac{dB}{dz}=\frac{dC}{dy};$$

indeed, we may successively consider z, y, and x as constant; and the differential, which has then only two terms, (since this supposition makes either $dz=0$, $dx=0$, or $dy=0$), must be nevertheless a complete differential, since the proposed one is so. It must therefore, in each of these cases, have the qualities of complete differentials with two variables.

It is easy, on the same principles, to find the necessary conditions of a greater number of variables.

Remark.

154. Let us suppose that Q is an unknown quantity composed of x, y, and constants, and that we know its differential $A\,dx$ found by regarding y as constant. If we wish to find the total differential of Q, we suppose it to be $A\,dx+B\,dy$; then B must be such that we should have $\frac{dA}{dy}=\frac{dB}{dx}$; therefore $dB=\frac{dA}{dy}dx$; we integrate, considering x only as variable, since x alone was made to vary in B.

We have $B = \int \frac{d\,A}{d\,y} d\,x$; whence $B\,d\,y = d\,y \int \frac{d\,A}{d\,y}\,d\,x$. Now, since $A\,d\,x$ is supposed to be the differential of Q, found by making x vary, we have $Q = \int A\,d\,x$, the integration being performed, considering x only as variable; therefore the complete differential of Q or of $\int A\,d\,x$ is $A\,d\,x + d\,y \int \frac{d\,A}{d\,y}\,d\,x$, or the integration $\int \frac{d\,A}{d\,y} d\,x$ must be performed considering y as constant

On Differential Equations.

155. When the proposed differential equation contains only two variables, x and y, and we have in one member the quantities x and $d\,x$, and in the other y and $d\,y$, the integration is reduced, for each member, to the rules given for differentials with a single variable.

Thus, the equation $a\,x^m\,y^n\,d\,x = b\,y^q\,x^r\,d\,y$, which may represent any differential equation with two terms, has its indeterminate quantities separated at once by dividing by y^n and by x^r, and becomes

$$a\,x^{m-r}\,d\,x = b\,y^{q-n}\,d\,y,$$

of which the integral is evidently

$$\frac{a\,x^{m-r+1}}{m-r+1} = \frac{b\,y^{q-n+1}}{q-n+1} + C.$$

156. But as it may happen that either one or both of the members of the differential equation thus separated may not be capable of integration algebraically, while the equation is nevertheless algebraical, or may at least be reduced to an algebraical form, it will be well to examine such of these cases as most frequently occur.

If, for example, in the preceding equation, we had $m - r = -1$, and $q - n = -1$, the differential equation would be reduced to $\frac{a\,d\,x}{x} = \frac{b\,d\,y}{y}$, and we can have the integral of each member only by means of logarithms; so that we may have $a\,l\,x = b\,l\,y + l\,C$.† But this equation may be rendered algebraical, by writing it

$$l\,x^a = l\,y^b + l\,C \text{ or } l\,x^a = l\,C\,y^b.$$

Now it is evident that if two logarithms are equal, the quantities to which they belong must be equal; wherefore $x^a = C\,y^b$, an algebraical equation.

157. If we had only $q - n = -1$, the differential equation would be $a\,x^{m-r}\,d\,x = \frac{b\,d\,y}{y}$, of which the integral is

$$\frac{a\,x^{m-r+1}}{m-r+1} = b\,l\,y + l\,C;$$

but we may give this equation an algebraical form by multiplying the

† It is allowable to suppose that the constant quantity is a logarithm.

first member by $l\,e$, e being the number whose logarithm is 1 (114); for we thereby produce no change in equation. We shall then have
$$\frac{a\,x^{m-r+1}}{m-r+1}\,l\,e = b\,l\,y + l\,C,$$
or, (making $m - r + 1 = p$), $l\,e^{\frac{a\,x^p}{p}} = l\,C\,y^b$, and consequently, $e^{\frac{a\,x^p}{p}} = C y^b$. Hereafter we shall always indicate by e the quantity whose logarithm is 1.

158. Let us take as a second example the equation
$$n\,d\,x = \frac{d\,z}{\sqrt{1-z^2}}.$$
The second member expresses (110) the element of that arc of a circle whose sine is z and radius 1. Whence z is the sine of $\int\frac{d\,z}{\sqrt{1-z^2}}$, that is, of $\int n\,d\,x$ or $n\,x + C$. We have therefore, for the integral, $z = \sin\,(n\,x + C)$. In like manner, from the equation
$$n\,d\,x = \frac{-\,d\,z}{\sqrt{1-z^2}},$$
we should infer $z = \cos\,(n\,x + C)$.

159. In the same way, since $\frac{d\,z}{1+z^2}$ (110) expresses the element of that arc of a circle whose radius is 1, and tangent z, if we had $n\,d\,x = \frac{d\,z}{1+z^2}$, we should conclude $z = \text{tang}\,(n\,x + C)$. And if we had $n\,d\,x = \frac{b\,d\,z}{a+f\,z^2}$; we should, in order to reduce it to the form of the preceding, make $z = m\,u$, m being a constant coefficient. We should then have $\frac{b\,m\,d\,u}{a+f\,m^2\,u^2}$; supposing therefore $f\,m^2 = a$, we should have $m = \sqrt{\frac{a}{f}}$, which would give
$$n\,d\,x = b\,\frac{\sqrt{\frac{a}{f}}\,d\,u}{a + a\,u^2},$$
whence we deduce $\frac{d\,u}{1+u^2} = \frac{n}{b}\,d\,x\,\sqrt{a\,f}$; wherefore u, or $\frac{z}{m}$, or
$$z\sqrt{\frac{f}{a}} = \text{tang}\left(\frac{n}{b}\,x\sqrt{a\,f} + C\right)$$
Therefore $z = \sqrt{\frac{a}{f}}\;\text{tang}\left(\frac{n}{b}\,x\sqrt{a\,f} + C\right)$.

160. In the expression sin $(n x + C)$, tang $(n x + C)$, which have just been found, $n x + C$ expresses the absolute length of the arc in parts of the radius 1. But as it is more convenient to employ the number of degrees, than the lengths themselves, it will be best, when we meet with such expressions, to estimate the arcs in degrees, which is easily done by dividing by the number of parts of the radius contained in a degree, that is, by 0,0174533 (125) or, which comes to the same thing, by multiplying by 57,2974166. Thus the sine of the arc whose length is b, and the sine of the arc which has a number of degrees expressed by $b \times 57,2974166$, are the same thing.

161. If we had $\frac{n\,dx}{\sqrt{1-x^2}} = \frac{dy}{\sqrt{1-y^2}}$, whose two members express the elements of two arcs which are to each other : : 1 : n, and whose sines are x and y; then, in order to integrate, we should make each member rational, by putting in the first, $\sqrt{1-x^2} = x\sqrt{-1} - z$, and in the second, $\sqrt{1-y^2} = y\sqrt{-1} - t$. The equation would be changed † (143) into $\frac{n\,dz}{z} = \frac{dt}{t}$, the integral of which is $n\,lz = l\,t + l\,C$, whence we have $C\,t = z^n$; and by substituting for t and z their values,

$$C\,(y\sqrt{-1} - \sqrt{1-y^2}) = (x\sqrt{-1} - \sqrt{1-x^2})^n,$$

which expresses generally the ratio of the sines x and y of two arcs which are multiples one of the other.

But in order to employ this equation, we must first determine the constant C. Now, supposing, as we may, that the two arcs have a common origin, then x and y must become zero at the same time. But in this case, the equation becomes

$$-C\sqrt{1} = (-\sqrt{1})^n, \text{ or } -C = (-1)^n;$$

now $(-1)^n$ is ± 1 or -1, according as n is even or odd; we have therefore $-C = -1$, and $C = \mp 1$, the upper sign being for the case in which n is even, and the lower, when n is odd. Therefore, finally

$$\mp\,(y\sqrt{-1} - \sqrt{1-y^2}) = (x\sqrt{-1} - \sqrt{1-x^2})^n.$$

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† By squaring, $1 - x^2 = -x^2 - 2xz\sqrt{-1} + z^2$, whence

$$x = \frac{z^2 - 1}{2z\sqrt{-1}}$$

$$dx = \frac{(z^2+1)\sqrt{-1}\,dz}{-2z^2};\ \sqrt{1-x^2} = \frac{-1-z^2}{2z} = -(z^2+1);$$

wherefore, $$\frac{n\,dx}{\sqrt{1-x^2}} = \frac{n\sqrt{-1}\,dz}{z}.$$

In the same manner, $\frac{dy}{\sqrt{1-y^2}} = \frac{\sqrt{-1}\,dt}{t}$, whence

$$\frac{n\sqrt{-1}\,dz}{z} = \frac{\sqrt{-1}\,dt}{t}, \text{ or } \frac{n\,dz}{z} = \frac{dt}{t}.$$

In each particular case, we may always make the imaginary quantities disappear; but the simplest way will be to transpose the whole to one side of the equation, and make some of the real quantities equal to zero; we shall then find the remaining equation to be divisible by $\sqrt{-1}$, and that it will be the same as that formed by making the sum of the real quantities equal to zero. If, for example, we make $n = 2$, we shall have

$$-y\sqrt{-1} + \sqrt{1-y^2} = -x^2 - 2x\sqrt{-1}\,.\,\sqrt{1-x^2} + 1 - x^2,$$

or

$$\sqrt{1-y^2} + 2x^2 - 1 + 2x\sqrt{-1}\,.\,\sqrt{1-x^2} - y\sqrt{-1} = 0.$$

Making then the sum of the real quantities equal to zero, we have

$$\sqrt{1-y^2} + 2x^2 - 1 = 0;$$

and the whole equation is reduced to

$$2x\sqrt{-1}\,.\,\sqrt{1-x^2} - y\sqrt{-1} = 0.$$

which being divided by $\sqrt{-1}$, gives

$$2x\sqrt{1-x^2} - y = 0, \text{ or } y = 2x\sqrt{1-x^2};$$

now, if we square this equation, and the equation

$$\sqrt{1-y^2} + 2x^2 - 1 = 0,$$

or rather $\sqrt{1-y^2} = 1 - 2x^2$, we shall have the same result.

We may find in the same manner the cosines and cotangents of multiple arcs. For then we should integrate $\frac{n\,dx}{1+x^2} = \frac{dy}{1+y^2}$ (111), by decomposing $1+x^2$ into $(1+x\sqrt{-1})(1-x\sqrt{-1})$, and $1+y^2$ into $(1+y\sqrt{-1})(1-y\sqrt{-1})$; we should then finish the work according to the rules laid down for rational fractions (133 and 136).

162. While we are upon this subject, we will make known a mode of expressing the sine and cosine of an arc, which may be of use.

Let $dx = \frac{dy}{\sqrt{1-y^2}}$ be the equation which expresses the relation between an arc x and its sine y. If we make $\sqrt{1-y^2} = y\sqrt{-1} - z$; we shall have $dx = \frac{-dz}{z\sqrt{-1}}$, or $\frac{dz}{z} = -dx\sqrt{-1}$, of which the integral is $lz = -x\sqrt{-1} + lC$, or $lz = -x\sqrt{-1}\,le + lC$ (157), which gives $z = Ce^{-x\sqrt{-1}}$; and substituting for z its value, we have $y\sqrt{-1} - \sqrt{1-y^2} = Ce^{-x\sqrt{-1}}$. With regard to the constant C, it may be determined by observing that the arc x and its sine must become zero at the same time. We shall therefore have $\sqrt{-1} = C$; wherefore $y\sqrt{-1} - \sqrt{1-y^2} = -e^{-x\sqrt{-1}}$; and consequently

$$\sqrt{1-y^2} = y\sqrt{-1} + e^{-x\sqrt{-1}};$$

squaring and reducing, we have

$$y=\frac{1-e^{-2x\sqrt{-1}}}{2\sqrt{-1}\,.\,e^{-x\sqrt{-1}}}=\frac{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{2\sqrt{-1}}$$

then, as y is the sine of x, we have

$$\sin x=\frac{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{2\sqrt{-1}}.$$

If, in the second member of the equation

$$\sqrt{1-y^2}=y\sqrt{-1}+e^{-x\sqrt{-1}},$$

we substitute for y the value just found, we shall have

$$\sqrt{1-y^2}\text{ or }\cos x=\frac{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{2}+e^{-x\sqrt{-1}}$$

$$=\frac{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{2};$$

therefore for the cosine we have

$$\cos x=\frac{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{2}.$$

But to return to the integration of equations.

163. When the indeterminate quantities are not separate in the proposed differential equation, it is best, before undertaking to separate them, to ascertain whether the equation be not integrable in the state in which it is. This may be known by examining (152) whether $\frac{dA}{dy}=\frac{dB}{dx}$, supposing $A\,dx+B\,dy=0$ to represent the equation. If this condition exists, we may integrate by the rules of 148.

164. It may however happen that this condition does not exist, while the equation is nevertheless integrable; which it may be sometimes rendered by multiplying it by some factor composed of x, y, and constants.

Let this factor be P. Then $AP\,dx+BP\,dy=0$ will be a complete differential. It is necessary therefore that

$$\frac{d(AP)}{dy}=\frac{d(BP)}{dx}.$$

The equation is thus reduced to finding for P a function of x, y, and constants, which may satisfy this equation. But as this investigation would lead too far, we shall only find P in the case when it contains only x and constants, or only y and constants. Suppose therefore that P is to contain only x, we shall have only

$$P\frac{dA}{dy}=B\frac{dP}{dx}+P\frac{dB}{dx};$$

whence we deduce

$$\frac{dP}{P}=\frac{\left(\frac{dA}{dy}-\frac{dB}{dx}\right)dx}{B},$$

we shall therefore easily find P, if $\frac{\frac{dA}{dy} - \frac{dB}{dx}}{B}$ is reduced to a function of x, as is necessary in order that P should be, as we suppose it, a function of x only.

We might also find the factor, if it were required that it should be composed of a function of x, multiplied or divided by a function of y of a known form.

165. By this means we integrate in general any equation of the form

$$Xy^q\,dy + X'\,y^{q+1}dx + X''\,y^r\,dx = 0,$$

X, X', X'' being any functions of x; q and r any exponents.

We might investigate whether it would not be made integrable by being multiplied by a factor of the form $P\,y^n$, P being a function of x, and n an indeterminate exponent, and should find that this may be done by supposing $n = -r$. But it is simpler to reduce immediately the whole equation to the form

$$y^{q-r}dy + Fy^{q-r+1}dx + F'\,dx = 0,$$

by dividing by X and y^r, and representing the quotients $\frac{X'}{X}$ and $\frac{X''}{X}$ by F and F'. In order then to integrate this, we suppose that P is the factor; P being a function of x. We shall then have

$$P\,y^{q-r}\,dy + FP\,y^{q-r+1}\,dx + F'P\,dx = 0.$$

Now if P is a function of x, FP will be so likewise, and $\int PF'dx$ will be reduced to the the integration of quantities with a single variable. It is only required then to render

$$P\,y^{q-r}dy + FP\,y^{q-r+1}\,dx$$

a complete differential, which requires that

$$\frac{d(P\,y^{q-r})}{dx} = \frac{d(FP\,y^{q-r+1})}{dy},$$

that is, that

$$y^{q-r}\frac{dP}{dx} = (q-r+1)\,y^{q-r}\,FP,$$

whence $$\frac{dP}{P} = (q-r+1)\,F\,dx;$$

integrating,

$$lP = \int(q-r+1)\,F\,dx = \int(q-r+1)\,F\,dx\,.\,le,$$

whence $$P = e^{\int(q-r+1)\,F\,dx}.$$

Substituting this value of P in the equation $P\,y^{q-r}dy +$ &c. and integrating, we have

$$\frac{y^{q-r+1}}{q-r+1}e^{\int(q-r+1)F\,dx} + \int F'\,y^{q-r+1}\,dx\,e^{\int(q-r+1)F\,dx} + C = 0.$$

We have added no constant quantity in the integration of the equation which gives P, because, there being no condition to determine it, we are at liberty to suppose it nothing.

Let us take an example. Suppose that we have to integrate

$$dy+\frac{a\,y\,dx}{x}+(b\,x^2+c\,x+f)\,dx=0.$$

Multiplying by P, we have

$$P\,dy+\frac{a\,y\,P\,dx}{x}+P\,(b\,x^2+c\,x+f)\,dx=0;$$

it is then necessary that

$$\frac{dP}{dx}=\frac{d\left(\frac{a\,y\,P}{x}\right)}{dy}=\frac{a\,P}{x};$$

whence $\frac{dP}{P}=\frac{a\,dx}{x}$; whence $lP=a\,lx$ or $P=x^a$.

Thus the equation becomes

$$x^a\,dy+a\,x^{a-1}\,dx\,y+b\,x^{a+2}\,dx+c\,x^{a+1}\,dx+f\,x^a\,dx=0,$$

of which the integral is

$$x^a\,y+\frac{b\,x^{a+3}}{a+3}+\frac{c\,x^{a+2}}{a+2}+\frac{f\,x^{a+1}}{a+1}+C=0.$$

166. The general equation just integrated frequently occurs; and the method we have given may be applied in many other cases. The following may be useful hereafter.

If we had two equations,

$$dx+a\,dy+(b\,x+c\,y)\,T\,dt=o,$$
$$k\,dx+a'\,dy+(b'\,x+c'\,y)\,T\,dt=o,$$

x, y, and t being variable; a, b, c, a', &c. constant, and T any function of t, we might reduce the integration of these two equations to the preceding method in the following manner. We multiply one of these, the first for example, by an indeterminate constant coefficient g, and adding it to the second, multiply the whole by a factor P, which we suppose to be a function of t; we have

$$(g\,P+k\,P)\,dx+(g\,a\,P+a'\,P)\,dy$$
$$+\left((g\,b\,P+b'\,P)\,x+(g\,c\,P+c'\,P)\,y\right)T\,dt=0.$$

We now suppose this equation to be an exact differential. We must have (153)

1st. $\frac{d\,(g\,P+k\,P)}{dt}=\frac{d\,(((g\,b\,P+b'\,P)\,x+(g\,c\,P+c'\,P)\,y)\,T)}{dx}$

2d. $\frac{d\,(g\,a\,P+a'\,P)}{dt}=\frac{d\,(((g\,b\,P+b'\,P)\,x+(g\,c\,P+c'\,P)\,y)\,T)}{dy}$;

3d. $\frac{d\,(g\,P+k\,P)}{dy}=\frac{d\,(g\,a\,P+a'\,P)}{dx}$. But P being a function of t, this last equation, in which t is considered constant, and therefore $d\,t=0$, is reduced to $0=0$. And the two others give

$$(g+k)\,\frac{dP}{dt}=(g\,b+b')\,PT,\text{ and }(g\,a+a')\,\frac{dP}{dt}=(g\,c+c')PT;$$

whence $\frac{dP}{P}=\frac{g\,b+b'}{g+k}\,T\,dt$, and $\frac{dP}{P}=\frac{g\,c+c'}{g\,a+a'}\,T\,dt$; wherefore putting these two values of $\frac{dP}{P}$ equal to each other, and dividing by

$T\,d\,t$, we have $\frac{g\,b+b'}{g+k}=\frac{g\,c+c'}{g\,a+a'}$, an equation in which g will be of the second degree, and which, being resolved, will give two values of g.

Supposing then g known, we shall easily find P, since the equation $\frac{d\,P}{P}=\frac{g\,b+b'}{g+k}\;T\,d\,t$ gives $P\,e=e\int\frac{g\,b+b'}{g+k}\;T\,d\,t$.

Now the equation $(g\,P+k\,P)\,d\,x+$ &c. being actually an exact differential, if we integrate it, we shall have

$$(g\,P+k\,P)\,x+(g\,a\,P+a'\,P)\,y+C=0\,;$$

therefore if g indicates the first value of g found from the above equation of the second degree, and we represent by g' the second value of g, and by P', the value which P takes by substituting g' for g, we shall also have $(g'\,P'+k\,P')\,x+(g'\,a\,P'+a'\,P')\,y+C'=0$, C' being a new constant. Indeed, there is no reason why we should employ one of these values of g rather than the other. And from these two equations it is easy to deduce the values of x and y, which will be expressed in terms of t and constants.

If the function T of t, which enters into the two equations, were different in each, we should proceed in the same manner, but should consider g as a function of t, and integrate as we should an equation of two variables, g and t.

If there were four variables, x, y, z, and t, expressed by three equations of the form

$$a\,dx+b\,d\,y+c\,dz+(e\,x+f\,y+hz)\;T\,d\,t=0,$$

and the function T were the same in each, we should integrate in the same manner, by multiplying the second and third by the indeterminate constant quantities g and g'; we should then add these two products to the first equation and multiply the whole by the factor P supposed to be a factor of t only. Then, supposing this new equation to be an exact differential, we should find (153) the equations from which to determine g, g' and P. The equation which gives g or that which gives g' will be of the third degree; from which we have three values for g, three corresponding values for g' and three for P. Changing the constant for each value of g, this will furnish three integrals, by means of which we may determine x, y and z in terms of t. We should proceed in the same way, if there were a greater number of variables, provided the equations were of the preceding form. The method would be the same, if there were one or more terms expressed in t, $d\,t$, and constants only.

167. And if we had in general any number m of equations comprehending $m+1$ variables, combined together in any way whatever, we should multiply the second, third, &c. to the last, respectively by g, g', g'', &c. supposed to be indeterminate functions of these variables, we should add them to the first equation, and multiply the whole by a factor P supposed to be a function of these variables, and then suppose the whole equation to be a complete differential. If, for example, we had the two equations

$$A\,dx+B\,d\,y+C\,dz=0,\;A'\,dx+B'\,d\,y+C'\,dz=0,$$

we should multiply the second by g; adding it to the first and multiplying the whole by P, we should have

$$P(A+A'g)dx+P(B+B'g)dy+P(C+C'g)dz=0.$$

Now, in order that this should be a complete differential, we must have (53)

$$\frac{d\,.\,P(A+A'g)}{dy}=\frac{d\,.\,P(B+B'g)}{dx};$$

$$\frac{d\,.\,P(A+A'g)}{dz}=\frac{d\,.\,P(C+C'g)}{dx};$$

$$\frac{d\,.\,P(B+B'g)}{dz}=\frac{d\,.\,P(C+C'g)}{dy}.$$

that is;

$$\frac{dP}{dy}(A+A'g)+P\frac{d(A+A'g)}{dy}$$
$$=\frac{dP}{dx}(B+B'g)+P\frac{d(B+B'g)}{dx};$$
$$\frac{dP}{dz}(A+A'g)+P\frac{d(A+A'g)}{dz}$$
$$=\frac{dP}{dx}(C+C'g)+P\frac{d(C+C'g)}{dx};$$
$$\frac{dP}{dz}(B+B'g)+P\frac{d(B+B'g)}{dz}=\frac{dP}{dy}(C+C'g)+P\frac{d(C+C'g)}{dy}.$$

If, from the two last equations, we deduce values of $\frac{dP}{dx}$ and $\frac{dP}{dy}$, and substitute them in the first, we shall have, after all reductions are made,

$$(C+C'g)\times\left(\frac{d(A+A'g)}{dy}-\frac{d(B+B'g)}{dx}\right)+(A+A'g)$$
$$\times\left(\frac{d(B+B'g)}{dz}-\frac{d(C+C'g)}{dy}\right)+(B+B'g)$$
$$\times\left(\frac{d(C+C'g)}{dx}-\frac{d(A+A'g)}{dz}\right)=0;$$

an equation not depending on P. We then find for g a function of x, y, and z, the most general possible and one which will satisfy this equation. Having found g, we find for P a function of x, y and z, which shall satisfy any two of the equations first found above, which, indeed, often requires a great deal of research, but which at least is always possible.

It is to be observed that if we had only a single equation, that is, if $A'=0, B'=0$, and $C'=0$, the last equation just found would be reduced to

$$C\left(\frac{dA}{dy}-\frac{dB}{dx}\right)+A\left(\frac{dB}{dz}-\frac{dC}{dy}\right)+B\left(\frac{dC}{dx}-\frac{dA}{dz}\right)=0,$$

which being an equation of the conditions between the coefficients

A, B, C, shows that in order that a differential equation with three variables, $A\,dx + B\,dy + C\,dz = 0$, may be integrable, even when multiplied by a factor, the coefficients $A, B\ C$ must have the relation indicated by the equation

$$C\left(\frac{dA}{dy} - \frac{dB}{dx}\right) + \&c. = 0.$$

When this condition is fulfilled, we determine the factor P, in such a manner as to satisfy two of the three equations

$$\frac{d(AP)}{dy} = \frac{d(BP)}{dx}, \frac{d(AP)}{dz} = \frac{d(CP)}{dx}, \frac{d(BP)}{dz} = \frac{d(CP)}{dy}.$$

We thus see what is to be done with a greater number of equations and a greater number of variables; and we may determine, in the same manner, what are the equations, in which it would be sufficient that g should be a constant, or a function of one or two of the variables, &c.

168 When the proposed differential equation does not come under any of the cases already given, we must endeavour to separate the indeterminate quantities. Sometimes the common rules of algebra are sufficient to effect this; at other times transformations are necessary. But there are many equations with regard to which it is difficult to determine what transformations are best.

The equation $ax^n\,dx + by^q x^n\,dx = y^k\,dy\,(e + fx^h)^r$ is separated immediately by division, and is the same as

$$(a + by^q)\,x^n\,dx = y^k\,dy\,(e + fx^h)^r,$$

which becomes $\dfrac{x^n\,dx}{(e + fx^h)^r} = \dfrac{y^k\,dy}{a + by^q}$, the integration of which is similar to that of a binomial quantity with a single variable.

And if we have

$$gx\,dx = ax^4 y\,dy + 2abx^2 y^3\,dy + ab^2 y^5\,dy;$$

we easily see that it may be written

$$gx\,dx = (x^4 + 2bx^2y^2 + b^2y^4)\,ay\,dy,$$

which may be written

$$gx\,dx = (x^2 + by^2)^2 \times ay\,dy.$$

Now with a little attention we see that the separation will succeed if we make $x^2 + by^2 = z$; indeed, we thus have $x^2 = z - by^2$ and $x\,dx = \frac{1}{2}\,dz - by\,dy$; by substitution,

$$\tfrac{1}{2}g\,dz - bgy\,dy = az^2 y\,dy;$$

an equation from which we deduce $\dfrac{\frac{1}{2}g\,dz}{bg + az^2} = y\,dy$, which is easily integrated.

169. As no general rules can be given for making the transformations, we shall confine ourselves to some very general cases in which the separation is known to succeed.

In general, the separation succeeds in all those equations with two variables, which are homogeneous, that is, in which the indeterminate quantities x and y have, in each term, either when combined or separate, the same sum of dimensions.

For, suppose $A\,dx + B\,dy = 0$ to be a homogeneous equation, and that we divide the whole by a power of x, whose exponent is equal to the number of the dimensions of the equation, it is easy to perceive that there will remain in A and B only powers of $\frac{y}{x}$ and constants; so that the equation will be $F\,dx + F'\,dy = 0$, F and F' being functions of $\frac{y}{x}$ and constants. This done, since

$$d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2},$$

we shall have $dx = \frac{-x^2}{y}\,d\left(\frac{y}{x}\right) + \frac{x}{y}\,dy$; if therefore we make $\frac{y}{x} = z$, we have $dx = -\frac{y\,dz}{z^2} + \frac{dy}{z}$, and substituting for $d\,x$, we have $-\frac{F\,y\,dz}{z^2} + \frac{F\,dy}{z} + F'\,dy = 0$, F and F' being now functions of z and constants. Now this equation gives

$$\frac{dy}{y} = \frac{F\,dz}{Fz + F'z^2},$$

an equation entirely separated, since F and F' contain no other variable than z.

For example, if we had $y^3\,dx + y^2\,x\,dy + b\,x^3\,dy = 0$, which is homogeneous, and the number of whose dimensions is 3, we should divide by x^3, we obtain

$$\frac{y^3}{x^3}\,dx + \frac{y^2}{x^2}\,dy + b\,dy = 0;$$

making, therefore, $\frac{y}{x} = z$, or $x = \frac{y}{z}$, we should have

$$dx = \frac{z\,dy - y\,dz}{z^2};$$

substituting in the proposed equation, it becomes

$$z^2\,dy - y\,z\,dz + z^2\,dy + b\,dy = 0,$$

whence $\frac{dy}{y} = \frac{z\,dz}{2z^2 + b}$, of which the integral is

$$l\,y = \tfrac{1}{4}\,l\,(2z^2 + b) + l\,C;$$

whence $\quad y = C\,(2z^2 + b)^{\frac{1}{4}}$, or $y^4 = C^4\,(2z^2 + b)$,

or finally $y^4 = C^4\left(2\,\frac{y^2}{x^2} + b\right)$, restoring in place of z its value $\frac{y}{x}$.

170. It would be of great use to be able to render equations homogeneous. But there is no general method for this purpose, and we are obliged to have recourse to transformations. Those which promise any success consist in making one of the variables, or a function of one, or even a function of two, equal to a function of a new variable with indeterminate exponents. These exponents are afterwards

determined by the condition that the transformed equation is to be homogeneous.

If, for example, we wish to find the cases in which the equation $a x^m d x + b y^n x^q d y + c y^k d y = 0$, to which form every equation of three terms may be reduced, may become homogeneous, we make $x = z^h$; we shall then have

$$a h z^{mh+h-1} d z + b y^n z^{qh} d y + c y^k d y = 0.$$

Now, in order that this be homogeneous, we must have $k = q h + n$, and $k = m h + h - 1$; whence

$$h = \frac{n+1}{m-q+1}, \text{ and } k = \frac{m n + q + n}{m - q + 1};$$

so that, if the exponents k, q, m, and n are such that this last equation may be true, we may render the equation homogeneous and consequently effect the separation.

171. In general, from not having any direct methods, we endeavour to reduce the proposed equations to other equations whose integration is known. We proceed thus, for example, with the particular equation $d y + a y^2 d x = b x^m d x$, known by the name of the equation of *Riccati*, and which can be integrated only for certain values of m.

If m were zero, it would be $d y + a y^2 d x = b\, d x$, which is separable, giving $\dfrac{d y}{b - a y^2} = d x$, which may be easily integrated.

But in order to integrate the equation when m has other values, we must endeavour to change it into another, in which $a y^2$ and b shall be multiplied by the same power of x; it will then become separable. The following is the method by which we find the values of m which allow of this transformation. We make $y = A x^p + x^q t$; whence $d y = p A x^{p-1} d x + q x^{q-1} t\, d x + x^q d t$; by substituting, we have

$$p A x^{p-1} d x + q x^{q-1} t\, d x + x^q d t + a x^{2q} t^2 d x + a A^2 x^{2p} d x$$
$$+ 2 a A x^{p+q} t\, d x = b x^m d x.$$

We suppose

$$p - 1 = 2 p, p A + a A^2 = 0, p + q = q - 1;\ q + 2 a A = 0;$$

and we have $p = -1$, $A = \dfrac{1}{a}$, $q = -2$; which changes the equation into $\quad x^{-2} d t + a x^{-4} t^2 d x = b x^m d x$,

or $\quad d t + a x^{-2} t^2 d x = b x^{m+2} d x$,

which will be separable if $m = -4$.

If we make, in this last, $t = \dfrac{1}{z}$, it will be changed into

$$d z + b x^{m+2} z^2 d x = a x^{-2} d x.$$

Making then $z = A' x^{p'} + x^{q'} t'$, and proceeding as before, we have

$$p' A' x^{p'-1} d x + q' x^{q'-1} t' d x + x^{q'} d t' + b x^{2q'+m+2} t'^2 d x$$
$$+ b A'^2 x^{2p'+m+2} d x + 2 b A' x^{p'+q'+m+2} t' d x = a x^{-2} d x.$$

If we suppose

$$2p' + m + 2 = p' - 1;\ p' A' + b A'^2 = 0;$$
$$q' + 2 b A' = 0,\ q' - 1 = p' + q' + m + 2;$$

we shall have

$$p' = -m - 3,\ A' = \frac{m+3}{b},\ q' = -2m - 6,$$

and $\qquad x^{-2m-6}\, dt' + b x^{-3m-10}\, t'^2\, dx = a x^{-2}\, dx,$

or $\qquad dt' + b x^{-m-4} t'^2\, dx = a x^{2m+4}\, dx,$

which will be separable if $-m - 4 = 2m + 4$, or if $m = -\frac{8}{3}$.

If we make $t' = \frac{1}{z'}$ and afterwards $z' = A'' x + x^{r''} t''$, and continue the same process as before, we shall find successively that the equation is separable when $m = -\frac{12}{5}$, $m = -\frac{16}{7}$, $m = -\frac{20}{9}$, &c.; that is, in general, when $m = \frac{-4r}{2r-1}$, r being a positive whole number.

Taking the above successive substitutions for t, t', t'', &c. we shall find that y has for its expression

$$y = A x^{-1} + \cfrac{x^{-2}}{A' x^{-m-3} + \cfrac{x^{-2m-6}}{A'' x^{-2m-5} + \cfrac{x^{-4m-10}}{A''' x^{-3m-7} + \&c.,}}}$$

continuing the process until the exponent of x in the first term of the last denominator shall be $-\overline{m+2}\cdot\overline{r-1} - 1$; and then the second term of these denominators will be $x^{-2m-4r-2}\, t$: t being a variable which, after the substitution of this value of y, is determined by the integration of the resulting equation, which is then separable. The only exception is the case in which $r = 1$, in which we have only to make $y = A x^{-1} + x^{-2} t$.

Let us resume the equation $dy + a y^2\, dx = b x^m\, dx$ and imagine that, instead of substituting at first $y = A x^p + x^q t$, as we have done above, we first make $y = \frac{1}{z}$, and then $z = A x^p + x^q t$, and proceed as above; we shall, in like manner, conclude, that we may effect the separation wherever $m = \frac{-4r}{2r+1}$, r being a positive whole number. And the value of y will be

$$y = \cfrac{1}{A x^{-m-1} + \cfrac{x^{-2m-2}}{A' x^{-2m-3} + \cfrac{x^{-4m-6}}{A'' x^{-3m-5} + \&c.}}}$$

Continuing in the same manner until the first term in x in the last denominator shall be of the power $-mr - 2r + 1$; and then the second term must be $x^{-2m-4r+2}\, t$.

We may reduce to the same case the equation

$x^q\, dy + a\, y^2\, x^n\, dx = b\, x^m\, dx$, by dividing by x^q, and then making $x^{n-q+1} = z$.

Such are the general methods to be employed when dx and dy do not exceed the first degree. As to equations containing different powers of dx and dy, as they cannot but be homogeneous with regard to dx and dy, we divide the whole by dx raised to a power equal to the sum of the dimensions of dx and dy; we then resolve the equation, considering $\frac{dy}{dx}$ as the unknown quantity. Then, as dx and dy will not be higher than the first degree, it will be perceived whether the preceding methods are applicable to the equation.

On Differential Equations of the second, third, and higher orders.

171. The liberty we have (19) in a differentiation, of considering any one of the first differences as constant, will contribute in very many cases to facilitate the integration. But as it may happen that in a differentiation, we have considered as constant the differential not most proper to facilitate the integration; we must begin by showing how we may reduce a differential equation in which some one difference is supposed constant, to another in which there shall be no constant. We may then suppose what we please constant. Let therefore $A\, dx^2 + B\, dx\, dy + C\, dy^2 + D\, ddy = 0$, be the equation with two variables and second differences, in which the first difference dx of one of the variables has been supposed constant. After having divided this equation by dx, we write it

$$A\, dx + B\, dy + \frac{C\, dy^2}{dx} + D\, d\left(\frac{dy}{dx}\right) = 0,$$

which is in fact the same, since, if we suppose dx constant, $d\left(\frac{dy}{dx}\right)$ is equal to $\frac{ddy}{dx}$. But if we do not consider dx as constant, then

$$d\left(\frac{dy}{dx}\right) = \frac{dx\, ddy - dy\, ddx}{dx^2};$$

whence the equation is changed into

$$A\, dx + B\, dy + \frac{C\, dy^2}{dx} + D\left(\frac{dx\, ddy - dy\, ddx}{dx^2}\right) = 0,$$

in which there is no difference constant.

Let

$$A\, dx^3 + B\, dx^2\, dy + C\, dy^2\, dx + D\, dy^3 + E\, dx\, ddy + F\, dy\, ddy + G\, d^3 y = 0,$$

be an equation with third differences, dx being always constant.

We divide by dx^2, and have

$$A\, dx + B\, dy + \frac{C\, dy^2}{dx} + D\frac{dy^3}{dx^2} + E\frac{ddy}{dx} + F\frac{dy}{dx} \cdot \frac{ddy}{dx} + G\frac{d^3 y}{dx^2} = 0,$$

which may be written

$$A\,dx + B\,dy + \frac{C\,dy^2}{dx} + \frac{D\,dy^3}{dx^2} + E\,d\left(\frac{dy}{dx}\right)$$

$$+ F\frac{dy}{dx}\,d\left(\frac{dy}{dx}\right) + G\,d\left(\left(\frac{1}{dx}\right)\,d\left(\frac{dy}{dx}\right)\right) = 0;$$

and, considering every quantity as variable in the differentiations here indicated, we shall have an equation in which there will be no longer any constant differential.

Let us apply these principles to an example. Let

$$dx^2\,dy - dy^3 = a\,dx\,ddy + x\,dx\,ddy$$

be an equation in which we have supposed dx constant. It cannot be immediately seen how this equation can be integrated; but if we render dx variable, by writing it

$$dx\,dy - \frac{dy^3}{dx} = (a\,dx + x\,dx)\,d\left(\frac{dy}{dx}\right),$$

we can, in the differentiation indicated in the second member, consider dy as constant, and we shall have

$$dx\,dy - \frac{dy^3}{dx} = -(a\,dx + x\,dx)\,\frac{dy\,ddx}{dx^2},$$

which becomes, when reduced,

$$dx^2 + x\,ddx + a\,ddx - dy^2 = 0,$$

of which the integral, as may be easily perceived, is

$$x\,dx + a\,dx - y\,dy + C\,dy = 0,$$

adding a constant $C\,dy$ of the same order as the integral. This equation, integrated anew, gives

$$\tfrac{1}{2}x^2 + ax - \frac{y^2}{2} + Cy + C' = 0.$$

172. Let us now examine equations with second differences and two variables. We give this name to those in which there is no difference exceeding the second order, to whatever power dx and dy may be otherwise raised.

We shall suppose one of the differences constant; but it will be easy to learn thence how to proceed if they were both variable.

Let then $A\,ddy + B = 0$ be the general equation which may represent any differential equation of the second order, with two variables x and y, and in which dx is constant. A and B are functions of x, y, dx, dy, and constants.

We write this equation in the form

$$A\,ddy + \left(\frac{B-k}{dy}\right)dy + \frac{k}{dx}\,dx = 0,$$

k being an unknown function of the same nature as A and B. We then multiply by P, supposed to be a function of x, y, dx, dy, and constants. We have

$$AP\,ddy + P\left(\frac{B-k}{dy}\right)dy + \frac{Pk}{dx}\,dx = 0,$$

which we suppose to be a complete differential.

This done, we have three differences, viz. ddy, dy, and dx. Considering these as the differences of so many different variables, we must have (153)

$$\frac{d(AP)}{dy}=d\frac{\left(P\left(\frac{B-k}{dy}\right)\right)}{ddy};\ d\frac{(AP)}{dx}=\frac{d\left(\frac{Pk}{dx}\right)}{ddy};$$

$$\frac{d\left(P\left(\frac{B-k}{dy}\right)\right)}{dx}=\frac{d\left(\frac{Pk}{dx}\right)}{dy}.$$

From these three equations, we may, by the process employed in *art.* 167, deduce an equation in which P shall not occur, and which may serve to determine k, taking for k a function the most general possible of x, y, dx, and dy, with indeterminate coefficients, which may be substituted in this equation. After which we may determine P, by taking, in like manner, a function of the same kind, and such as to satisfy two of these three equations. But we may simplify this investigation, by confining it to finding for P a function of x, y, dx, and dy, which shall satisfy two of the equations.

The first two of the three equations just found, give

$$\frac{d(AP)}{dy}=-\frac{1}{dy^2}P(B-k)+\frac{1}{dy}\cdot\frac{d(P(B-k))}{ddy},$$

or

$$\frac{d(AP)}{dy}=-\frac{1}{dy^2}P(B-k)+\frac{1}{dy}\frac{d(PB)}{ddy}-\frac{1}{dy}\frac{d(Pk)}{ddy},$$

and $$\frac{d(AP)}{dx}=\frac{1}{dx}\frac{d(Pk)}{ddy}.$$

Substituting, in the last equation but one, the value of $\frac{d(Pk)}{ddy}$, deduced from the last, we have

$$\frac{d(AP)}{dy}=-\frac{1}{dy^2}P(B-k)+\frac{1}{dy}\frac{d(PB)}{ddy}-\frac{dx}{dy}\frac{d(AP)}{dx};$$

whence we have

$$P(B-k)=dy\frac{d(PB)}{ddy}-dx\,dy\frac{d(AP)}{dx}-dy^2\frac{d(AP)}{dy};$$

whence it will be easy to find k when P is known.

From this last equation we deduce

$$Pk=PB-dy\frac{d(PB)}{ddy}+dx\,dy\frac{d(AP)}{dx}+dy^2\frac{d(AP)}{dy};$$

substituting the value of $P(B-k)$ and of Pk in the equation

$$\frac{d(AP)}{dy}=\frac{d\left(P\frac{(P-k)}{dy}\right)}{ddy},$$

and in the equation

$$\frac{d\left(\frac{P(B-k)}{dy}\right)}{dx} - \frac{d\left(\frac{Pk}{dx}\right)}{dy},$$

and we shall have

$$\frac{d(AP)}{dy} = \frac{d\left(\frac{d(PB)}{ddy} - dx\frac{d(AP)}{dx} - dy\frac{d(AP)}{dy}\right)}{ddy},$$

and

$$\frac{d\left(\frac{d(PB)}{ddy} - dx\frac{d(AP)}{dx} + dy\frac{d(AP)}{dy}\right)}{dx}$$

$$= \frac{d\left(\frac{PB}{dx} - \frac{dy}{dx}\frac{d(PB)}{ddy} + dy\frac{d(AP)}{dx} + \frac{dy^2}{dx}\frac{d(AP)}{dy}\right)}{dy}.$$

The question is therefore reduced to finding for P a function of x, y, dx, dy, and constants, which shall satisfy these two equations. But although this be always possible, it is not always easy; for which reason, we shall leave this general investigation, and examine some equations more limited, but still very much extended.

We first observe however that it is easy, by the principles just given, to ascertain whether the equation is integrable in its present state, we have only to suppose $P = 1$, when, if the equation be integrable, it will satisfy the two following equations:

$$\frac{dA}{dy} = \frac{d\left(\frac{dB}{ddy}\right) - dx\left(\frac{dA}{dy}\right) - dy\left(\frac{dA}{dy}\right)}{ddy},$$

and

$$\frac{d\left(\frac{dB}{ddy} - dx\frac{dA}{dx} - dy\frac{dA}{dy}\right)}{dx}$$

$$= \frac{d\left(\frac{dB}{dx} - \frac{dy}{dx}\frac{dB}{ddy} + dy\frac{dA}{dx} + \frac{dy^2}{dx}\frac{dA}{dy}\right)}{dy}.$$

This is general, whatever may be the differential equation of the second order, dx being constant.

173. Let it now be proposed to integrate the equation

$$G\,dx^2 + H\,dx\,dy + K\,dy^2 + L\,ddy = 0,$$

in which the factor P which is necessary to render the equation integrable, need only be a function of x, y, and constants. It is supposed, moreover, that G, H, K, and L, contain neither dx nor dy, but are only functions of x, y, and constants.

If we compare this equation with the general equation

$$A\,ddy + B = 0,$$

we have $A = L$, and $B = G\,dx^2 + H\,dx\,dy + K\,dy^2$. Substituting in the two equations found above, in order to determine P, and observing that we have supposed P, G, H, K, and L to contain

neither dx nor dy, we shall have $\frac{d(PL)}{dy} = KP$; and a second equation, which, after we have substituted in it for $\frac{d(PL)}{dy}$, its value KP, is reduced to

$$\frac{d\left(PHdx + KPdy - dx\frac{d(PL)}{dx}\right)}{dx} = \frac{d\left(PGdx + dy\frac{d(PL)}{dx}\right)}{dy}.$$

But since $\frac{d(PL)}{dy} = KP$,

we have

$$\frac{d(KP)}{dx} = \frac{d\left(\frac{d(PL)}{dy}\right)}{dx} = \frac{\frac{dy}{dy}\,d\,\frac{d(PL)}{dx}}{dy};$$

and consequently

$$\frac{d(KPdy)}{dx} = \frac{dy\,d\left(\frac{d(PL)}{dx}\right)}{dy};$$

wherefore our second equation is reduced, after having divided each member by dx, to

$$\frac{d(PH)}{dx} - \frac{dd(PL)}{dxdx}\dagger = \frac{d(PG)}{dy}.$$

This equation and the equation $\frac{d(PL)}{dy} = KP$ are the equations which we have now to operate on, in order to integrate the proposed equation.

We observe now, that in this last equation, y only is to be considered as variable. This being fixed, performing the differentiation indicated, and deducing the value of $\frac{dP}{P}$ we have

$$\frac{dP}{P} = \frac{K}{L}dy = \frac{dL}{L};$$

taking therefore the integral, y only being considered as variable, since the differentiation was performed on that supposition, we shall have

$$lP = \int\frac{K}{L}dy - lL + lX.$$

We add the quantity lX for a constant, by which we understand a

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† By the expression $\frac{dd(PL)}{dxdx}$, is to be understood that we ought to differentiate PL, making x variable, and divide afterwards by dx, and then differentiate the result, making x again variable, and divide by dx.

function of x and constants; because x has been supposed constant in the differentiation.

From this equation we deduce $P=\frac{X}{L}\,e^{\int\frac{K}{L}dy}$. If we substitute this value of P in the equation $\frac{d(PH)}{dx}$ — &c., and then divide by $e^{\int\frac{K}{L}dy}$, we shall have an equation from which to determine X. But as X must be a function of x, it follows, that in order that the proposed equation be integrable by the multiplication of a factor composed only of x, y, and constants, all the y's in this equation must disappear.

Let us suppose, for example, that we have the equation

$$2y\,dx^2+(2x+3yx)\,dx\,dy+2x^2\,dy+x^2\,y\,ddy=0,$$

which in its present form is not integrable. We have

$$L=x^2y,\ G=2y,\ H=2x+3yx,\ K=2x^2;$$

therefore

$$P=\frac{X}{x^2y}\,e^{\int\frac{2dy}{y}}=\frac{Xe}{x^2y}\,l\,y^2=\frac{X}{x^2y}\,y^2=\frac{Xy}{x^2}.$$

Substituting this value of P, and those of L, G, H, &c. in the equation $\frac{d(PH)}{dx}$ &c., we shall have, after transposition,

$$-\frac{4Xy}{x^2}+\frac{2y\,dX}{x\,dx}-\frac{2Xy}{x^2}+\frac{3y^2\,dX}{x\,dx}-\frac{3Xy^2}{x^2}-\frac{y^2\,ddX}{dx^2}=0,$$

making the sum of the terms affected by y equal to zero, and then dividing one of the equations by y and the other by y^2, we shall have after making all the reductions,

$$\frac{dX}{X}=\frac{3dx}{x},\ \text{and}\ -x^2\,ddX+3x\,dX\,dx-3X\,dx^2=0.$$

The first gives $X=x^3$; and this value, substituted in the second, satisfies it; we have therefore $X=x^3$, and consequently

$$P=\frac{x^3y}{x^2}=xy.$$

If we now go back to the value of Pk, found (172), we shall have

$$Pk=2xy^2\,dx+3x^2\,y^2\,dx\,dy,$$

and $\qquad P(B-k)=2x^2\,y\,dx\,dy+2x^3\,y\,dy^2;$

so that the equation, brought to the general form (172), becomes

$$x^3y^2ddy+(2x^2y\,dx+2x^3y\,dy)dy+(2xy^2dx+3x^2y^2\,dy)\,dx=0.$$

In order to integrate, we follow the rule given (148); we first take $x^3\,y^2\,ddy$, and consider dy only as variable, which gives $x^3\,y^2\,dy$. Differentiating this quantity, considering all as variable, and subtracting from the equation, there remains

$$(2x^2\,y\,dx)\,dy+(2xy^2\,dx)\,dx.$$

We integrate the first of these terms, considering y only as variable,

and we shall have $x^2 y^2 dx$, the differential of which, taking x and y as variable, subtracted from the preceding remainder, leaves nothing; whence the integral is, when a constant is added,

$$x^3 y^2 dy + x^2 y^2 dx + C dx = 0$$

We may take as a second example, the equation

$$2dx^2 + (3x + y + 2) dy dx + 2x dy^2 + (x^2 + xy) ddy = 0;$$

which is integrated in the same manner. We shall find that X must be equal to x, and $H = xy$.

174. If, after the substitution of the value of P, in the equation $\frac{d(PH)}{dx}$, &c., all the y's disappear of themselves, the equation which must give X is then a differential of the second order; whence it appears, that the method is, in this case, of no use. But it must be observed that the equation which will then be obtained, will be of the form

$$A dx^2 + BX dx^2 + C dX dt + E ddX = 0,$$

A, B, C, E being functions of x and of constant quantities. Now, in order to integrate this equation, we must write it thus

$$AP' dx^2 + BP'X dx^2 + (C - k')P' dx dP + k'P' dP dx + EP' d dX = 0.$$

We now suppose that, P' and k' being functions of x only, the last four terms taken together form an exact differential; then the first term, being a function of x, will be readily integrated.

The equations which result from this supposition, are

$$\frac{d(EP')}{dx} = \frac{d(k'P'dX + BP'Xdx)}{ddX},$$

$$\frac{d(EP')}{dX} = \frac{d((C' - k')P'dx)}{ddX},$$

$$\frac{d(k'P'dX + BP'Xdx)}{dX} = \frac{d[(C - k')P'dX]}{dx},$$

and

$$\frac{d[(C - k')P'dx]}{dx} = \frac{d(BP'Xdx)}{dX}.$$

These four equations are reduced to the two following (from the consideration that k', P', A, B, &c. do not contain P),

$$\frac{d(EP')}{dx} = k'P', \text{ and } BP' = \frac{d[(C - k')P']}{dx}.$$

Deducing from each of these equation, the value of $\frac{dP'}{P'}$, and putting one of these values equal to the other, we shall have, after making all reductions,

$$E dk' + (C - k') dE - k'(C - k') dx + BE dx - E dC = 0,$$

a differential equation of only the first order, and on which depends the value of X, and consequently the integral of the proposed equation. Supposing, therefore, that k' has been determined by means of this equation, we may easily obtain P', by means of the equation

$$k'P' = \frac{d(EP')}{dx} \text{ which gives } \frac{dP'}{P'} = \frac{k'dx}{E} - \frac{dE}{E},$$

and consequently

$$P' = \frac{H}{E}\, e^{\int \frac{k'dx}{E}},$$

H being a constant quantity. When the values of k' and P' are found, we may find X, by substituting the values k' and P' in the equation

$$AP'dx^2 + BP'Xdx^2 + (C - k')P'dxdX + k'P'dXdx + EP'ddX = 0,$$

and integrating. Now as this equation cannot fail of being a complete differential, we have for its integral

$$dx\int AP'dx + Xdx\int BP'dx + dX\int k'P'dx + Ldx = 0,$$

L being a constant quantity. This is easily integrated by what has been already laid down (165). We may therefore find X whenever we can find k'; whence it may be laid down as universally true, that whenever nothing is wanting to make the equation

$$Gdx^2 + Hdxdz + Kdy^2 + Lddy^2 = 0,$$

an exact differential, but a factor composed of x, y, and constants, this equation will be always reducible to a differential equation of the first order, whatever may be the value of G, H, K, L.

But if, after the substitution of the value of P, in the equation $\frac{d(PH)}{dx}$ &c., the equation still contains y, which cannot be made to disappear, without subjecting the coefficients G, H, K, L, to certain conditions, we conclude that the factor P must also contain x and dy; we must then have recourse to the general method (172).

We might proceed in the same manner to ascertain in what cases any other differential equation of the second order, of a known form, may be integrated by multiplication by a factor composed of x, y, and constants, or of x, dy, dx, and constants, or of y, dx, and constants, &c.

175. With regard to differential equations of the third order, if we suppose them to be represented generally by $Ad^3y + B = 0$, A and B being functions of x, y, dx, dy, ddy, and constants; if we suppose, moreover, that P is the factor composed of x, y, dx, ddy, and constants, which will render it integrable, we may write it thus

$$APd^3y + P\frac{B-k}{ddy}ddy + P\frac{k-h}{dy}dy + \frac{Ph}{dx}dx = 0.$$

Then the following equations must be true.

$$\frac{d(AP)}{ddy} = \frac{d\left(P\frac{(B-k)}{ddy}\right)}{d^3y};\quad \frac{d(AP)}{dy} = \frac{d\left(P\frac{(k-h)}{dy}\right)}{dy^3};$$

$$\frac{dAP)}{dx} = \frac{d\left(\frac{Ph}{dx}\right)}{d^3y};\quad \frac{d\left(P\frac{(B-k)}{ddy}\right)}{dy} = \frac{d\left(P\frac{(k-h)}{dy}\right)}{ddy};$$

$$\frac{d\left(P\frac{(B-k)}{ddy}\right)}{dx}=\frac{d\left(\frac{Ph}{dx}\right)}{ddy};$$

$$\frac{d\left(P\frac{(k-h)}{dy}\right)}{dx}=\frac{d\left(\frac{Ph}{dx}\right)}{dy}.$$

By means of these equations k, h, and P may be determined. But we shall carry this investigation no farther.

The process would be similar for differential equations of still higher orders.

176. It may be observed in conclusion, 1°. That when one of the two finite variables is wanting in an equation, it may be always reduced to an equation of a lower degree, by making $dy=p\,dx$, p being a new variable.

177. 2°. That the general equation

$d^n y+a\,d^{n-1}y\,dx+b\,d^{n-2}y\,dx^2+\&c.\ldots+hy\,dx^n+X\,dx^n=0,$

a, b, &c. being constants, X a function of x and constants, and dx being constant, may always be easily integrated by a method similar to that employed above, for the equation

$$A\,dx^2+BX\,dx^2+C\,dX\,dx+E\,ddX=0.$$

To this end, it must be written

$$P\,d^n y+P(a-k)\,d^{n-1}y\,dx+Pk\,d^{n-1}y\,dx$$
$$+P(p-k')\,a^{n-2}y\,dx^2+Pk'\,d^{n-2}y\,dx^2+\&c.$$
$$\ldots\ldots+Phy\,dx^n+PX\,dx^n=0,$$

P being the factor which will render the equation integrable, and which we suppose to be a function of x; and k, k', &c. indeterminate constants.

We shall suppose that the terms, taken two and two, beginning with the first, form an exact differential. This supposition will give the equations necessary for determining P, k, k', &c. Having put the values of $\frac{dP}{P}$ equal to each other, we shall have equations in terms of k, k', &c. by means of which k may be determined by an equation of the degree n. The value of k being found, we may easily find that of k', k'', &c. and that of P will be obtained by integrating, which will be without difficulty done. Then for each value of k, we shall have a particular integral, observing to add to each a different constant. From $n-1$ of these equations we may deduce the values of dy^{n-1}, dy^{n-2}, &c. and by substituting them in the last, we shall obtain the value of y in terms of x.

178. 3°. If we should have several equations in which the differences were not multiplied together, except that they were multiplied by the constant difference, and in which the variables should not exceed the first degree, nor be multiplied together, we might integrate them by multiplying the second, third, &c. each by a constant factor p, p', &c. adding them to the first, and multiplying the whole by a factor P, supposed to be a function of the variable whose difference is constant. We should then decompose the terms affected by the differences of the same variable, as in the preceding equation.

If, for example, we had

$$a\,ddz + b\,ddy + (c\,dz + e\,dy)\,dx + (fz + gy)\,dx^2 = 0,$$

and

$$a'\,ddz + b'\,ddy + (c'\,dz + e'\,dy)\,dx + (f'z + g'y)\,dx^2 = 0,$$

by multiplying the second by p, adding it to the first and multiplying the whole by P, we should have

$$P(a + a')\,ddz + P(c + c'p)\,dz\,dx + P(f + f'p)\,z\,dx^2$$
$$+ P(b + b'p)\,ddy + P(e + e'p)\,dy\,dx + P(g + g'p)\,dx^2 = 0.$$

We should then decompose $c + c'p$, into $c + c'p - k$ and k; and $e + e'p$ also, into $e + e'p - k'$ and k'. Then supposing that the terms, taken two and two, form exact differentials, we should have the equations necessary to determine k, k', and P. The equation in terms of x will rise generally, to the degree $2n$, which will furnish $2n$ integrals, by means of which we may eliminate all the differences and obtain the equations in terms of z and x, of y and x, &c.

179. 4°. If the equations were still more general, p, p', &c., as well as P, might be considered as functions of all the variables and their differences, and these functions might be determined by the condition of the total equation being a complete differential.

NOTES.

Note referred to in Art. 95.

SINCE an equation to a conic section is always of the second degree, and since the most general equation of this degree may in every case be reduced to the form

$$b\,t^2 + c\,u\,t + e\,u^2 + f\,t + g\,u + h = 0,$$

it follows that we may always make a conic section pass through five given points, provided that these points, taken three and three, are not in the same straight line, a conic section never meeting a straight line in more than two points.

Suppose A, B, C, D, E (*fig.* 56) to be five given points, having this condition. If we refer these points to the line AD, which joins two of them, by drawing the lines BF, CH, EG, at a given angle or perpendicular to AD, then the distances AF, BF; AG, GE; AH, HC; AD, which are considered as known, may be regarded as the abscissas and ordinates of a curve line. Now we may always suppose that this curve line has for its equation

$$b\,t^2 + c\,u\,t + e\,u^2 + f\,t + g\,u + h = 0;$$

for, let $AF = n$; $BF = m$; $AG = n'$; $GE = m'$; $AH = n''$; $CH = m''$; $AD = n'''$; then it is evident that, 1st. For the point A, we shall have $u = 0$, and $t = 0$, which reduces the equation to $h = 0$. 2d. For the point B we shall have $u = n$, and $t = m$, which changes the equation into $b\,m^2 + c\,m\,n + e\,n^2 + f\,m \ldots\ldots g\,n = 0$, since $h = 0$. 3d. For the point E we shall have $u = n'$, and $t = m'$, and consequently $b\,m'^2 + c\,m'\,n' + e\,n'^2 + f\,m' + g\,n' = 0$. 4th. For the point C, we shall in the same maner find

$$b\,m''^2 + c\,m''\,n'' + e\,n''^2 + f\,m'' + g\,n'' = 0.$$

5th. For the point D, where $t = 0$, and $u = n'''$, we shall have

$$e\,n'''^2 + g\,n''' = 0, \text{ or } e\,n''' + g = 0.$$

Now as these four equations contain all the quantities c, e, f, g, of the first degree, it will be easy to find their values; then, by substituting them in the equation

$$b\,t^2 + c\,u\,t + e\,u^2 + f\,t + g\,u + h = 0,$$

or rather $\quad b\,t^2 + c\,u\,t + e\,u^2 + f\,t + g\,u = 0,$

since $h = 0$, we shall have the value of c, e, f, g in quantities wholly known, and the equation will be divisible by b. It will then be easy

to construct the curve and to determine whether it be an ellipse, hyperbola, parabola, or circle. If only four points were given, one of the coefficients would be arbitrary; this would give the power of imposing, at pleasure, one condition; if only three points were given, two conditions might be imposed, and so on.

We distinguish lines by the degree of their equation. Thus the straight line, whose equation is of the first degree, is a line of the first order. The conic sections are lines of the second order. It will be seen, therefore, that the above method may be used to determine the equation of a line of the third order, which may be made to pass through as many points less one as the general equation of this order, with two indeterminates, has different terms. The same may be said of the higher orders.

The method under consideration will serve to connect, by an approximate and simple law, several known quantities, the law of which is very compounded or unknown. Suppose, for example, that three quantities are known, which may be represented by the lines *CB*, *ED*, *GF* (*fig.* 57), and that thse quantities depend upon three others *AB*, *AD*, *AF*. It is proposed to find a quantity *HI* intermediate between the first, or situated near them, and which is derived from *AH* after the manner in which *CB*, *ED*, &c. is derived from *AB*, *AD*, &c. This question may be satisfied in an infinite number of ways by taking an equation with two indeterminates, u and t, having at least as many different terms as it contains such quantities as *CB*, *ED*, *GF*. But among all these different ways, that which is the most readily applicable to the different purposes to be answered by this method, is to regard the line *IH* as the ordinate, and the line *AH* as the abscissa passing through the given points *C*, *E*, *G*, &c., and which has for this equation

$$t = a + bu + c u^2 + \&c.,$$

by taking as many terms as there are quantities or points *C*, *E*, *G*; then by supposing as above, that u is equivalent to *AB*, t will be equal to *CB*; and u being equivalent to *AD*, t will be equal to *DE*; and u being equivalent to *AF*, t will be equal to *GF*, and so on; we have thus as many equations for determining a, b, c, &c., as we have points. The values of a, b, c, &c., being determined, if we substitute them in the equation $t = a + b\,u + c\,u^2 + \&c.$, we shall have an equation in which every thing will be known except u and t, accordingly if we put for u the known distance *AH*, which answers to the quantity sought *HI*, we shall have the corresponding value of t, or *HI*. If we would imitate the perimeter *ABCDEF* (*fig.* 58), we should let fall perpendiculars from a certain number of

the points of this curve upon a determinate line HZ, since by the method just laid down, we can determine the equation of a curve that shall pass through all these points, and in which t being of the first degree, u will be of the degree denoted by the number of these points less one; then this equation will serve to determine intermediate perpendiculars, approaching so much the nearer to the true ones, according as we take in the first place a greater number of points A, B, C, D, &c. See *Bézout's Algebra*, Art. 411.

Note 2.

General Demonstration of the Binomial Formula.

In Lacroix's Algebra, (*art.* 136, & *seq.*) is given a demonstration of the binomial theorem for the case of positive integral exponents. The following demonstration of the same formula for the case of exponents of any value whatever, integral or fractional, positive or negative, is taken from the Elements of Algebra by Bourdon, (*art.* 202, & *seq.*) It may first be observed that the binomial $x+a$ may be put under the form $x\left(1+\frac{a}{x}\right)$; whence it follows that

$(x+a)^m = x^m\left(1+\frac{a}{x}\right)^m = x^m(1+z)^m$, supposing $\frac{a}{x}=z$.

If then it can be shown that the formula

$$(1+z)^m = 1+mz+m\frac{m-1}{2}z^2+m\frac{m-1}{2}.\frac{m-2}{3}z^3+\ldots$$

holds true, whatever be the value of m, it will follow, if we substitute $\frac{a}{x}$ for z, and multiply by x^m, that

$$(x+a)^m = x^m\left(1+m\frac{a}{x}+m\frac{m-1}{2}.\frac{a^2}{x^2}+\ldots\right)$$

$$= x^m+m\,a\,x^{m-1}+m\frac{m-1}{2}a^2x^{m-2}+\ldots,$$

and that this last formula must be considered as true.

Now it has been shown that when m is any whole number (*Alg.* 136, *p.*)

$$(1+z)^m = 1+mz+m\frac{m-1}{1}z^2+m\frac{m-1}{2}.\frac{m-2}{3}z^3+\text{\&c.}$$

We now proceed to inquire from what algebraical expression the series

$$1+mz+m.\frac{m-1}{2}z^2+m\frac{m-1}{2}.\frac{m-2}{3}z^3+\text{\&c.}$$

is derived, when m is a positive fractional number $\frac{p}{q}$. If we indicate this unknown expression by y, we have the equation

$$y = 1 + mz + m\frac{m-1}{2}z^2 + m\frac{m-1}{2}.\frac{m-2}{3}.z^3 + \&c. \quad (1)$$

If m' be another positive fractional exponent, we shall in like manner have

$$y' = 1 + m'z + m'\frac{m'-1}{2}z^2 + m'\frac{m'-1}{2}.\frac{m'-2}{3}z^3 + \&c. \quad (2)$$

If now we multiply the equations (1) and (2) together, member by member, we shall obtain $y\,y'$ for the first member. As to the second member, it follows from the laws of multiplication (*Alg. art.* 31), that *the form of a product does not depend on the particular value of the letters which enter into its factors;* consequently, the above product must have the same form as in the case where m and m' are positive whole numbers. But in this case we have

$$1 + mz + m\frac{m-1}{2}z^2 + \ldots = (1+z)^m,$$

$$1 + m'z + m'\frac{m'-1}{2}z^2 + \ldots = (1+z)^{m'};$$

whence

$$\left(1 + mz + m\frac{m-1}{2}z^2 + \ldots\right)\left(1 + m'z + m'\frac{m'-1}{2}z^2 + \ldots\right)$$
$$= (1+z)^{m+m'} = 1 + (m+m')z + (m+m')\frac{(m+m'-1)}{2}z^2 + \ldots$$

therefore, the formula just obtained belongs equally to the case in which m and m' have any value whatever, in which case we have

$$y\,y' = 1 + (m+m')z + (m+m')\frac{(m+m'-1)}{2}z^2 + \ldots \quad (3)$$

Let m'' be a third positive fractional exponent, we shall have

$$y'' = 1 + m''z + m''\frac{m''-1}{2}z^2 + \ldots.$$

Multiplying these equations together, member by member, we obtain

$$y\,y'\,y'' = 1 + (m+m'+m'')z + (m+m'+m'')\frac{(m+m'+m''-1)}{2}z^2 \ldots$$

Universally let there be a number q of exponents m, m', m'', m''' .. (q being the denominator of m or $\frac{p}{q}$,) we shall have

$$y\,y'\,y''\,y''' \ldots = 1 + rz + r\frac{r-1}{2}z^2 + r\frac{r-1}{2}.\frac{r-2}{3}z^3 + \ldots \quad (4),$$

r representing the sum of the exponents $m + m' + m'' + m''' \ldots$

If we now suppose $m = m' = m'' = m''' \ldots$ in which case we have $r = m + m + m + \ldots\ldots = m\,q$, the equation (4) becomes

$$y^q = 1 + m\,q.\,z + m\,q.\frac{m\,q-1}{2}z^2 + m\,q.\frac{m\,q-1}{2}.\frac{m\,q-2}{3}z^3 + \ldots$$

Now we have by supposition, $m = \frac{p}{q}$, whence $m\,q = p$; wherefore

$$y^q = 1 + p\,z + p\frac{p-1}{2}z^2 + p\frac{p-1}{2}.\frac{p-2}{3}z^3 + \ldots;$$

but p is a whole number, so that the second member of this equation is the development of $(1+z)^p$; which gives us the equation $y^q = (1+z)^p$, whence $y = (1+z)^{\frac{p}{q}} = (1+z)^m$; therefore we conclude that

$$(1+z)^m = 1 + m\,z + m\frac{m-1}{2}z^2 + m\frac{m-1}{2}.\frac{m-2}{3}z^3 + \ldots$$

m being any positive fractional number whatever.

In order to demonstrate the truth of the formula for the case in which m is negative, either whole or fractional, it will be sufficient to suppose, in equation (3) which is formed of equations (1) and (2), $m' = -m$, which reduces equation (3) to $y\,y' = 1$ (since $m + m' = 0$), from which we deduce $y = \frac{1}{y'}$. But since, by hypothesis, m is negative, m' or the value $-m$ must be positive, and we have also $y' = (1+z)^{m'} = (1+z)^{-m} = \frac{1}{(1+z)^m}$,

whence

$$y = \frac{1}{(1+z)^{m'}} = (1+z)^{-m'} = (1+z)^m,$$

and consequently

$$(1+z)^m = 1 + m\,z + m\frac{m-1}{2}z^2 + \ldots.$$

Note 3.

On the Method of Indeterminate Coefficients. From Bourdon's Algebra, (Art. 208 and 209.)

To give an idea of this method, let it be proposed to develope the expression $\frac{a}{a' + b'\,x}$ in a series proceeding according to the ascending powers of x. The development may evidently take place

since $\frac{a}{a'+b'x}$ may be reduced to the form $a\,(a'+b'x)^{-1}$; and by applying the binomial formula we may find the development sought. Let us therefore suppose it performed, and that

$$\frac{a}{a'+b'x}=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+\ldots \text{(1)}$$

$A, B, C, D, E, \ldots$ being functions of a, a', b', but independent of x, and which, as their value is to be determined, are called ***indeterminate coefficients.***

In order to determine these coefficients, we multiply both members of the equation (1) by $a'+b'x$, arrange the terms according to the powers of x, transpose the term a, and we have

$$0=\left\{\begin{array}{l|l|l|l|l} Aa'+Ba' & x+Ca' & x^2+Da' & x^3+Ea' & x^4+\ldots \\ -a+Ab' & +Bb' & +Cb' & +Db' & \end{array}\right. \text{(2)}$$

We now observe that, if we suppose suitable values to be given to $A, B, C, D, \ldots$ equation (1) must be verified, whatever value is given to x; the same is true of equation (2).

Now if we suppose $x=0$, the latter equation becomes $0=Aa'-a$; whence we obtain as the value of A, $A=\frac{a}{a'}$; and if A is equal to $\frac{a}{a'}$, when $x=0$, it must preserve the same value, whatever be the value of x, since by supposition, A is independent of x: thus, whatever may be the value of x, equation (2) is reduced to

$$0=\left\{\begin{array}{l|l|l|l} Ba' & x+Ca' & x^2+Da' & x^3+\ldots \\ +Ab' & +Bb' & +Cb' & \end{array}\right. \text{ or dividing by } x$$

$$0=\left\{\begin{array}{l|l|l} Ba'+Ca' & x+Da' & x^2+\ldots \\ +Ab'+Bb' & +Cb' & \end{array}\right. \text{(3).}$$

Now as this equation must also be verified whatever value is given to x, we make $x=0$, and the equation becomes $Ba'+Ab'=0$, from which we deduce $B=-\frac{Ab'}{a'}$, or $B=\frac{a}{a'}\times-\frac{b'}{a'}=-\frac{ab'}{a'^2}$.

As B must retain the same value, whatever may be that of x, we suppress in equations (3), the first term $Ba'+Ab'$ which this value of B renders equal to 0, and divided by x, and we have

$$0=\left\{\begin{array}{l|l|l} Ca'+Da' & x+Ea' & x^2+\ldots \\ +Bb'+Cb' & +Db' & \end{array}\right.$$

Again making $x=0$, this equation becomes $Ca'+Bb'=0$, whence we deduce $C=-\frac{Bb'}{a'}$, or $C=-\frac{ab'}{a'^2}\times-\frac{b'}{a'}=\frac{ab'^2}{a'^3}$.

By the same process we should find $Da'+Cb'=0$, whence

$$D=-\frac{Cb'}{a'} \text{ or } D=\frac{ab'^2}{a'^3}\times-\frac{b'}{a'}=-\frac{ab'^3}{a'^4};\ \&c.$$

Here it is easy to perceive that any coefficient is formed from that which precedes it, by multiplying that coefficient by $-\frac{b'}{a'}$; in this way we have

$$\frac{a}{a'+b'x}=\frac{a}{a'}-\frac{ab'}{a'^2}x+\frac{ab'^2}{a'^3}x^2-\frac{ab'^3}{a'^4}x^3+\frac{ab'^4}{a'^5}x^4\ldots.$$

The fundamental principle of this method of indeterminate coefficients is this. *If an equation of the form* $0=M+Nx+Px^2+Qx^3+\ldots$ ($M, N, P, \ldots$ being coefficients independent of x), *may be verified, whatever value is given to x, each separate coefficient must necessarily be equal to* 0.

Indeed, since these coefficients are independent of x, if we obtain their values, by making particular suppositions with regard to x, the value obtained will still belong to them, whatever value is given to x. Now if we make $x=0$, we find $M=0$, and by dividing by x, the equation is reduced to

$$0=N+Px+Qx^2+\ldots;$$

if we make again in this new equation $x=0$, we find $N=0$, and by dividing by x, the equation is reduced to $0=P+Qx+\ldots$ &c. We have therefore separately

$$M=0, N=0, P=0, Q=0\ldots;$$

and by this means we obtain as many equations as there are coefficients to be determined.

The application of this method requires that one be previously acquainted with the mode of developing with reference to the exponents of x. In ordinary cases the development may proceed according to the different ascending powers of x, but sometimes the expression must be separated into factors, before it is developed. The expression $\frac{1}{3x-x^2}$, for example, must be put under the form $\frac{1}{x}\times\frac{1}{3-x}$, and we must then suppose

$$\frac{1}{x}\times\frac{1}{3-x}=\frac{1}{x}(A+Bx+Cx^2+Dx^3+\ldots).$$

Note 4.

Of the Methods which preceded and in some Measure supplied the Place of the Infinitesimal Analysis.

There are several methods of resolving questions analogous to that of the infinitesimal analysis; and although there are none which unite the same advantages, it may not be the less curious to examine the different points of view under which this theory may be regarded.

On the Method of Exhaustions.

This is the method which the ancients made use of in their difficult researches, and especially in the theory of curved lines and curved surfaces, and in the estimatiom of the areas and solidities contained by them. As they admitted no demonstrations which were not perfectly rigorous, they would not allow themselves to consider curves as polygons of a great number of sides. But when they wished to discover the properties of a curve, they considered it as the fixed term or limit to which the inscribed and circumscribed polygons continually approach, and as nearly as we please according as the number of sides is increased. In this way they *exhausted,* as it were, the space comprehended between these polygons and the curve; which circumstance doubtless procured for this mode of proceeding the name of *the method of exhaustions.*

As the polygons thus made use of were known figures, their continual approximation to the curve, so as finally to differ from it by less than any given quantity, led to the knowledge of the properties of the curves under examination.

But geometricians were not satisfied with thus inferring or divining, as it were, the properties of curves; they would have them verified incontestibly; this they effected by proving that any supposition contrary to the existence of these properties led necessarily to some contradiction. This kind of demonstration was called *reductio ad absurdum.*

By this means, having first ascertained that the areas of similar polygons are to each other as the squares of their homologous lines, they inferred that circles of different radii are to each other as the squares of their radii. This is the second proposition of the 12th book of Euclid, and the 287th article of Legendre's Geometry. Analogy led them to this conclusion, by imagining regular polygons of the same number of sides, to be inscribed in the given circles. For, as upon

increasing to any degree the number of these sides, their areas remain as the squares of the radii of the circumscribed circles, they easily perceived that the same thing must hold of the circles, to which these polygons continually approached. But this was not enough. It was necessary rigorously to demonstrate that this is true in fact, and this they did by showing that every contrary supposition necessarily leads to an absurdity.

In this manner the ancients demonstrated that the solidities of spheres are to each other as the cubes of their diameters, that a cone is the third part of the cylinder of the same base and altitude; propositions which are contained in the fourth section of Legendre's Geometry.

By means of inscribed and circumscribed figures, they also demonstrated the properties of curved surfaces and of the solidities contained by them. The law of continuity led them to the conclusion, and the conclusion was verified by a reductio ad absurdum.

In this manner Archimedes demonstrated that the convex surface of a right cone is equal to a circle which has for its radius the mean proportional between the side of the cone and the radius of the base, that the whole surface of a sphere is equal to that of four of its great circles, and that the surface of a spherical zone is equal to the circumference of a great circle multiplied by the altitude of the zone.

It was also by a reductio ad absurdum, that the ancients extended to incommensurable quantities, the relations which they had discovered between commensurable quantities. This method is certainly very beautiful, and of very great value. It carries with it the character of the most perfect evidence, and never permits its object to be lost sight of; it was the method of invention among the ancients, and is to this day very useful, because it exercises the judgment, accustoms one to rigorous exactness in demonstrations, and includes the germ of the infinitesimal analysis. It is true that it requires some considerable exertion of mind; but is not the power of profound meditation indispensable to all those who would penetrate into a knowledge of the laws of nature? And is it not necessary early to form the habit, provided we do not sacrifice too much time to its attainment?

On observing with attention the processes made use of in the method of exhaustions, we perceive that there is a great resemblance between them and those used in the infinitesimal analysis. In each, auxiliary quantities are employed, always containing some thing arbitrary in their statement; from considering the properties of these quantities, inferences are drawn with regard to the un-

known properties of the curve or other quantity in question. The auxiliary quantities are then omitted, and the desired result remains freed from every thing uncertain or arbitrary.

But "though few things more ingenious than this method have been devised, and though nothing could be more conclusive than the demonstrations resulting from it, yet it labored under two very considerable defects. In the first place, the process by which the demonstration was obtained was long and difficult; and, in the second place, it was indirect, giving no insight into the principle on which the investigation was founded. Of consequence, it did not enable one to find out similar demonstrations, nor increase one's power of making more discoveries of the same kind. It was a demonstration purely synthetical, and required, as all indirect reasoning must do, that the conclusion should be known before the reasoning is begun." *

In the hands of Newton, this doctrine made great progress towards perfection. His *prime and ultimate ratios,* are precisely the ratios made known by the gradual approximation of the auxiliary quantities to the quantities whose properties are sought.

By this theory Newton extended the principles of the method of exhaustions, and he simplified its processes by freeing it from the necessity of having its results verified by a reductio ad absurdum, and by showing that these results are sufficiently established by the accuracy of the mode employed to obtain them.

Newton thus expresses himself in the conclusion of the view he gives of his theory. "These lemmas are premised to avoid the tediousness of deducing perplexed demonstrations *ad absurdum,* according to the method of the ancient geometers." †

This great man advanced this doctrine far more considerably, by reducing this very method of prime and ultimate ratios to an algorithm, in his method of fluxions. By means of this calculus he introduced into algebraical analysis, not only these prime and ultimate ratios, but their terms taken separately, which was a modification of great importance, on account of the new means of transformation which it furnished. Newton, however, did not enjoy this glory alone; he shared it with Leibnitz, who had the advantage of publishing his algorithm first, and who, being powerfully seconded by other celebrated geometricians, associated with him, advanced his method far more rapidly than the method of fluxions was brought forward during the same time.

* Playfair. View of the Progress of Mathematical and Physical Science.

† Scholium to Lemma XI. Sec 1, Book I. of the Principia.

On the Method of Indivisibles.

Cavalieri was the forerunner of the inventers of the infinitesimal analysis, and opened the way for them by his *Geometry of Indivisibles.* He was led to this by a tract of Kepler, on the measure of solids, in which he introduced, for the first time, the consideration of infinitely great and infinitely small quantities.

In the method of indivisibles, solids are considered as composed of an infinite number of parallel surfaces, surfaces as composed of an infinite number of lines, and lines of an infinite number of points.

These suppositions are certainly absurd, and should be employed with caution. They are rather to be considered as means of abbreviation, by the help of which, we obtain readily and easily, in many cases, what would require long and laborious processes by the method of exhaustions. For example.

Let AB (*fig.* 59,) be the diameter of a semicircle AGB, $ABFD$ the circumscribed rectangle, CG the radius perpendicular to DF; draw the two diagonals CD, CF, and, through any point m of the straight line AD draw the straight line $m\,n\,p\,g$ perpendicular to CG, cutting the circumference in the point n, and the diagonal CD in the point p.

Conceive the whole figure to turn about CG as an axis; the quadrant ACG will generate the solid volume of the hemisphere whose diameter is AB, the rectangle $ADGC$ will generate the circumscribed right cylinder, the right angled isosceles triangle CGD will generate a right cone, having for its altitude and for the radius of its base the equal lines CG, DG; finally, the three straight lines $m\,g$, $n\,g$, $p\,g$, will each generate a circle, whose centre will be at the point g.

Now the first of these three circles is the element of the cylinder, the second is the element of the hemisphere, and the third that of the cone.

Moreover, the areas of these circles being as the squares of their radii, and these three radii being the hypothenuse and sides of a right angled triangle, (since $C\,g = p\,g$, and $m\,g = C\,n$), it is evident that the first of these circles is equal to the sum of the two others: that is, the element of the cylinder is equal to the sum of the corresponding elements of the hemisphere and cone, and, as it is the same with all the other elements, it follows that the total solidity of the cylinder is equal to the sun of the total solidity of the hemisphere and the total solidity of the cone.

But we know that the solidity of the cone is one third of that of the cylinder, therefore the solidity of the hemisphere is two thirds;

that is, the solidity of the entire sphere is two thirds of the solidity of the circumscribed cylinder, as was discovered by Archimedes.

Cavalieri professed to consider his method as only a corollary of the method of exhaustions, but confessed that he could not give a rigorous demonstration of it. The great geometers who succeeded him soon caught the spirit of this method, and it was in vogue with them, until the discovery of the new mode of calculation. It was to this that Pascal and Roberval owed the success of their profound researches on the cycloid. The former of these distinguished authors thus expresses himself in relation to this subject.

"For this reason, I shall not hesitate hereafter to make use of the language of indivisibles—*the sum of the lines, the sum of the planes;* I shall not hesitate to use the expression, *the sum of the ordinates,* which seems not to be geometrical, to those who do not understand the doctrine of indivisibles, and who think it is sinning against geometry, to express a plane by an indefinite number of lines. But this comes of their not understanding it, since nothing is meant thereby, but the sum of an indefinite number of rectangles, each formed by an ordinate and one of the small equal portions of the diameter, the sum of which is certainly a plane. So that when we speak of the *sum of an indefinite multitude of lines,* we have always reference to a certain straight line, by the equal and indefinite portions of which they are multiplied."

This passage is remarkable, not only as it shows that these geometers knew how to appreciate rightly the merit of the method of indivisibles, but still more, as it proves that the notion of mathematical infinity, in the very sense which is at this day given it, was not unknown to them. For it is evident, from the passage just cited, that Pascal attached to the word *indefinite* the same signification which we attach to the word *infinite,* that he called by the word *small,* what we understand by *infinitely small,* and that he neglected, without hesitation, these small quantities by the side of finite quantities: for we see that he considered as simple rectangles the trapezoids or small portions of the area of the curve, which are comprehended between two consecutive ordinates, neglecting, consequently, the small mixtilineal triangles, which have for their bases the difference of these ordinates. No one, however, has dared to reproach Pascal with want of rigor.

We shall conclude the notice of this method with one or two examples.

Common algebra teaches us how to find the sum of any number of terms taken in the series of natural numbers, the sum of their

squares, that of their cubes, &c., and this knowledge furnishes to the geometry of indivisibles, the means of estimating the area of a great number of rectilineal and curvilinear figures, and the solidities of a great number of bodies.

Let there be, for example, a triangle; from its vertex let fall a perpendicular upon the base; divide this perpendicular into an infinite number of equal parts, and through each of the points of division draw a straight line parallel to the base, and terminating in the two other sides of the triangle.

According to the principles of the geometry of indivisibles, we may consider the area of the triangle as the sum of all the parallels which are regarded as its elements; now, by a well known property of triangles, these straight lines are proportional to their distances from the vertex; therefore, the altitude being supposed to be divided into equal parts, these parallels increase in an arithmetical progression, of which the first term is zero.

But in every arithmetical progression, whose first term is zero, the sum of all the terms is equal to the last multiplied by half the number of terms. Now, in this case, the sum of the terms is represented by the area of the triangle, the last term by the base, and the number of terms by the altitude. Therefore the area of every triangle is equal to the product of its base by half its altitude.

Again; let there be a pyramid: From its vertex let fall a perpendicular upon the base, divide this perpendicular into an infinite number of equal parts, and through each point of division, let a plane pass parallel to the base of this pyramid.

According to the principles of the geometry of indivisibles, the intersection of each of these planes, with the solidity of the pyramid, will be one of the elements of this solidity, which will be simply the sum of all these elements.

But by the properties of the pyramid, these elements are to each other as the squares of their distances from the vertex. Calling the base of the pyramid B, its altitude A, any one of the elements just mentioned b, its distance from the vertex a, and the solidity of the pyramid S, we shall have

$$B : b :: A^2 : a^2,$$

therefore

$$b = \frac{B}{A^2}\,a^2.$$

Therefore S, which is the sum of all these elements, is equal to the constant quantity $\frac{B}{A^2}$ multiplied by the sum of the squares a^2;

and since the distances a increase in an arithmetical progression, (the first term of which is zero, and the last A,) that is, as the natural numbers from 0 to A, the quantities a^2 will represent the squares of these distances from 0 to A^2.

Now common algebra shows us that the sum of the squares of the natural numbers from 0 to A^2 inclusively is

$$\frac{2A^3+3A^2+A}{6}$$

But the number A in this case being infinite, all the terms which follow the first in the numerator disappear, by the side of this first term; therefore this sum of the squares is reduced to $\frac{1}{3}A^3$.

Multiplying therefore this value by the constant quantity $\frac{B}{A^2}$ found above, we shall have for the solidity sought

$$S=\tfrac{1}{3}BA;$$

that is, the solidity of a pyramid is the third part of the product of its base by its altitude.

By a similar process it is proved that, generally, the area of any curve which has for its equation

$$a\,y^m = x^n,$$

is $\frac{m}{m+n}.XY$; Y representing the last ordinate, X the corresponding abscissa, m, n, any exponents, whether integral, fractional, positive, or negative.

Thus the method of indivisibles supplies in some respects the place of the integral calculus; it may be regarded as corresponding to the integration of simple quantities, and this certainly was a great discovery for the time of Cavalieri.

On the Method of Indeterminate Quantities.

It seems to me that Descartes, by his method of indeterminates, approached very near to the infinitesimal analysis, or rather, that the infinitesimal analysis is only a fortunate application of the method of indeterminates.

Let there be an equation with only two terms

$$A+B\,x=0,$$

in which the first term is constant, and the second susceptible of being rendered as small as we please. According to what has been shown, (note 3,) this equation cannot hold unless the terms A and $B\,x$ are each, separately, equal to zero. Therefore we may lay it down as a general principle, and as an immediate consequence of the method of indeterminates, that *if the sum or difference of two pre-*

tended quantities is equal to zero, and if one of the two may be supposed as small as we please, while the other contains nothing arbitrary, these two pretended quantities will be each separately equal to zero.

This principle alone is sufficient to resolve by common algebra all questions falling under the infinitesimal analysis. The respective processes of the two methods, simplified as they may be, are absolutely the same. The whole difference consists in the mode of considering the question. The quantities which in the one are *neglected* as infinitely small, are *understood* in the other, though considered as finite, since it is demonstrated that they must eliminate themselves, that is, that they must destroy each other in the result of the calculation.

Indeed, it is easy to see that this result can only be an equation of two terms, of which each is separately equal to zero. We may therefore beforehand, suppose to be understood in the course of the calculation, all those quantities which belong to that one of these two terms, of which no use is to be made. Let us apply this theory of indeterminates to some examples.

Let it be proposed to prove that the area of a circle is equal to the product of its circumference by half the radius; that is, that calling this radius R, the ratio of the circumference to radius, π, and consequently the circumference πR, the surface of the circle S,

$$S = \tfrac{1}{2} \pi R^2.$$

In order to do this, we inscribe in the circle a regular polygon, then double successively the number of its sides, until the area of the polygon differs as little as we please from that of the circle. At the same time, the perimeter of the polygon will differ as little as we please from the circumference of the circle, and the straight line drawn from the centre to the middle of a side, as little as we please from the radius R. Then the surface S, will differ as little as we please from $\frac{1}{2} \pi R^2$; consequently, if we make

$$S = \tfrac{1}{2} \pi R^2 + \varphi,$$

the quantity φ, if it is not zero, may at least be supposed as small as as we please. Now we put this equation under the form

$$(S - \tfrac{1}{2} \pi R^2) - \varphi = 0,$$

an equation of two terms, the first of which contains nothing arbitrary, while the second, on the contrary, may be supposed as small as we please; then, by the theory of indeterminates, each of these terms separately is equal to 0; thus we have

$$S - \tfrac{1}{2} \pi R^2 = 0, \text{ or } S = \tfrac{1}{2} \pi R^2;$$

which was to be demonstrated.

that is, we have exactly

$$d\left(\frac{B}{3A^2}x^3+C\right)=\frac{B}{A^2}x^2\,dx+\frac{B}{A^2}(3x\,dx^2+dx^3);$$

taking then the exact sum of each member, we have

$$\left(\frac{B}{3A^2}x^3+C\right)=\text{sum}\,\frac{B}{A^2}x^2\,dx+\text{sum}\,\frac{B}{A^2}(3x\,dx^2+dx^3),$$

or, transposing,

$$\text{sum}\,\frac{B}{A^2}x^2\,dx=\left(\frac{B}{3A^2}x^3+C\right)-\text{sum}\,\frac{B}{A^2}(3x\,dx^2+dx^3).$$

Substituting in equation (1), we shall have exactly

$$S=\left(\frac{B}{3A^2}x^3+C\right)-\left(\text{sum}\,\frac{B}{A^2}(3x\,dx^2+dx^3)-\text{sum}\,\varphi\right),$$

an equation in which the last term only contains arbitrary quantities, and may be supposed as small as we please. For the sake of conciseness, make this term φ'; the equation will become, by transposition,

$$\left(S-(\frac{B}{3A^2}x^3+C)\right)-\varphi'=0,$$

an equation of which, by the principles of the method of indeterminates, each term taken separately is equal to zero, whence

$$S=\frac{B}{3A^2}x^3+C.$$

In order to determine C, we have only to make $x=0$, then we have $S=0$, whence $C=0$; wherefore the equation is reduced to

$$S=\frac{B}{3A^2}x^3,$$

that is, the solidity of the pyramid from the vertex to the altitude x is $\frac{Bx^3}{3A^2}$; in order therefore to obtain the whole solidity of the pyramid, we have only to suppose $x=A$, which will give

$$S=\tfrac{1}{3}BA.$$

This solution, as may be easily seen, is no other than that which would be obtained by the processes of the infinitesimal analysis, by neglecting nothing, and the common infinitesimal analysis is only an abbreviation of these processes, since we neglect only the quantities φ, φ', which, in the result of the calculation, fall only on that one of the two equations into which the quantities are decomposed, of which no use is made. Now, what the infinitesimal analysis neglects may, by a simple fiction under the name of quantities infinitely small, be understood, in order to preserve the rigor of geometry during the whole course of the calculation. We thus see that the method of indeterminates furnishes a rigorous demonstration of

the infinitesimal calculus, and that it affords at the same time the means of supplying the place of it, if we choose, by common algebra. It were to be wished, perhaps, that this course had been pursued in arriving at the differential and integral calculus; it would have been as natural as the method that was actually taken, and would have prevented all difficulties.

Of the Method of Prime and Ultimate Ratios, or of Limits.

The method of prime and ultimate ratios, or of limits, has also its origin in the method of exhaustions, of which it is, properly speaking, only a development and simplification. We owe this useful improvement to Newton, and it is in his book of the Principia that it is to be studied. It will be sufficient for our purpose to give here a succinct idea of it.

When two quantities are supposed to approach each other continually, so that their ratio or the quotient arising from dividing the one by the other, differs less and less, and finally as little as we please, from unity, these two quantities are said to have for their ultimate ratio a ratio of equality.

In all cases, when we suppose different quantities to approach respectively and simultaneously other quantities which are considered as fixed, until they differ respectively and at the same time, as little as we please, the ratios which these fixed quantities have to each other are the *ultimate ratios* of those which are supposed to approach them respectively and simultaneously, and these fixed quantities themselves are called *limits* or *ultimate values* of the quantities so approaching.

These values and ratios are called ultimate values and ultimate ratios respectively, or prime values and prime ratios, of the quantities to which they are referred, according as the variables are considered as approaching to or receding from the quantities, considered as fixed, which serve as their limits.

These limits, or quantities considered as fixed, may, however, be variable, as would, for example, be the coördinates of a curve; that is, they may not be given by the conditions of the question, but be only determined by the subsequent hypotheses on which the calculation rests. Thus, for example, though the coördinates of a curve are comprehended among the quantities called variable, because they are not of the number of the data; yet, if I propose a problem to be resolvsd respecting any particular curve, as that of drawing a tangent to it, it will be necessary, in order to establish my reasonings, and calculation, that I should begin by assigning deter-

minate values to these coördinates, and that I should continue to the end of the process to regard them as fixed. Now these quantities, considered as fixed, are comprehended, as well as the data of the problem, among the quantities called limits.

These limits are the quantities whose ratio is sought. Those which are supposed gradually to approach them are only auxiliary quantities, which are interposed to facilitate the expression of the conditions of the problem, but which must necessarily be eliminated in order to obtain the result sought.

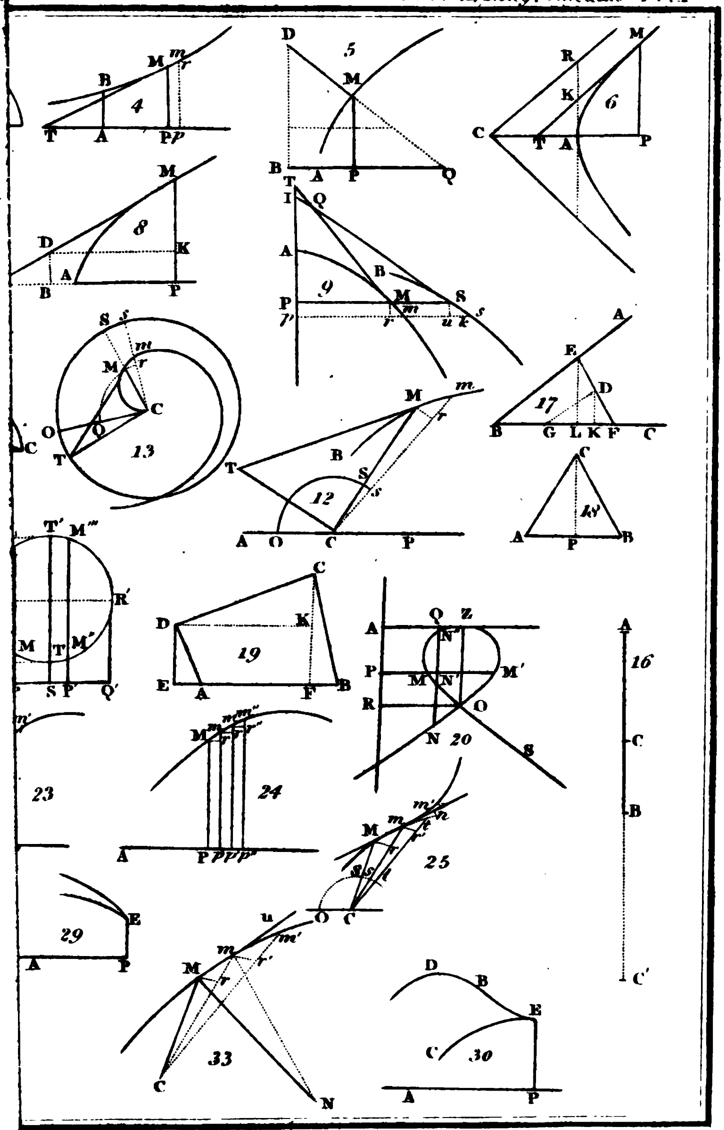
We thus see the analogy which must exist between the theory of prime and ultimate ratios and the infinitesimal method. What, in the latter, are called infinitely small quantities, are evidently the same as the difference between any quantity and its limits; and those quantities whose ultimate ratio is a ratio of equality, are those which, in the infinitesimal analysis, are said to differ infinitely little from each other.

The principal difference between that which is called the method of limits and the infinitesimal method, consists in this, that in the former, we can admit into the process of calculation only the limits themselves, which are always definite quantities, while in the latter we may also employ the variable quantities, which are supposed to approach them continually, as well as the difference between them and their limits. This gives the infinitesimal method more means of varying its expressions and its algebraical transformations, without introducing the least difference in the rigor of the processes.

The property thus obtained by the infinitesimal method renders it susceptible of a new degree of perfection still more important, which is the power of being reduced to a particular algorithm. For these differences between the variable quantities and their limits, are what we distinguish by the name of *differentials* of their limits, and the simplification to which the admission of these quantities into the calculation gives occasion, are precisely what gives the infinitesimal analysis its importance.

The method of limits or of prime and ultimate ratios, is nevertheless far superior, for the facility of its processes, to the simple method of exhaustions; since it is at least freed from the necessity of a reductio ad absurdum for each particular case, by far the most difficult operation in the method of exhaustions; while, by the other method, it is sufficient, in order to prove the equality of any two quantities, to show that they are both limits of the same third quantity.

If this method were always as easy in its application as the common infinitesimal analysis, it might seem preferable, for it would have the advantage of leading to the same results by a process direct and always clear. But it must be confessed that the method of limits is subject to a considerable difficulty which does not belong to the common infinitesimal analysis. It is this, that as the infinitely small quantities are always connected by pairs, and cannot be separated from each other, we cannot introduce into the combinations, properties which belong to each of them separately, nor subject the equations in which they occur to all the transformations which would be necessary in order to eliminate them.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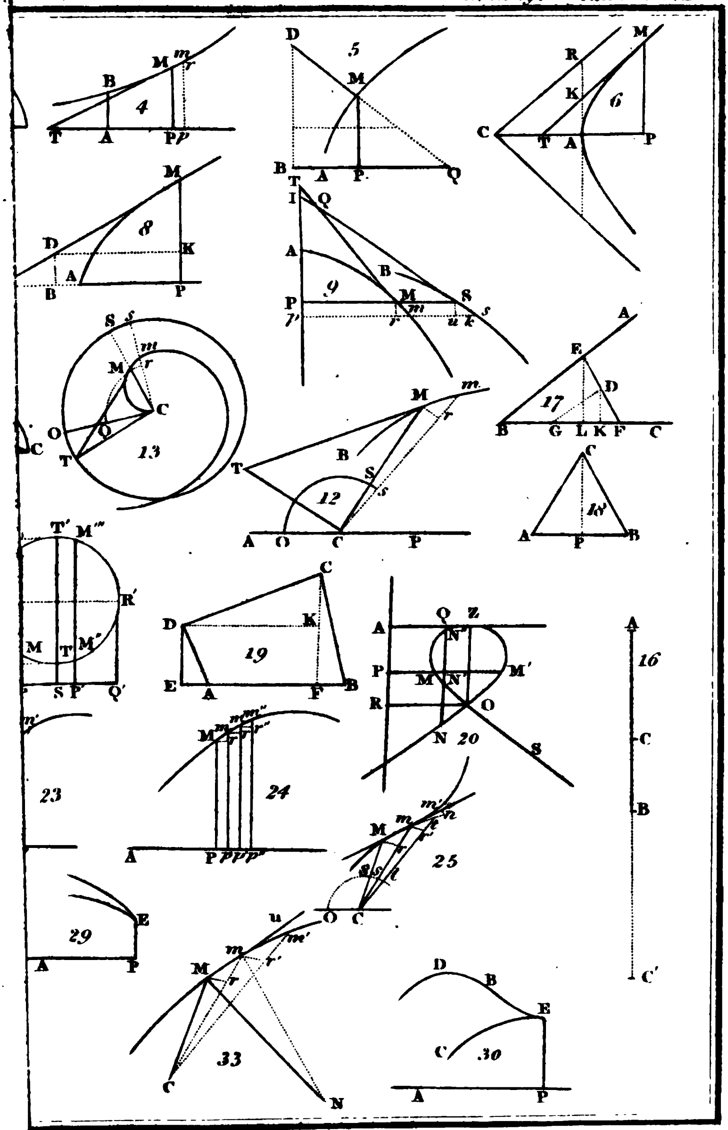

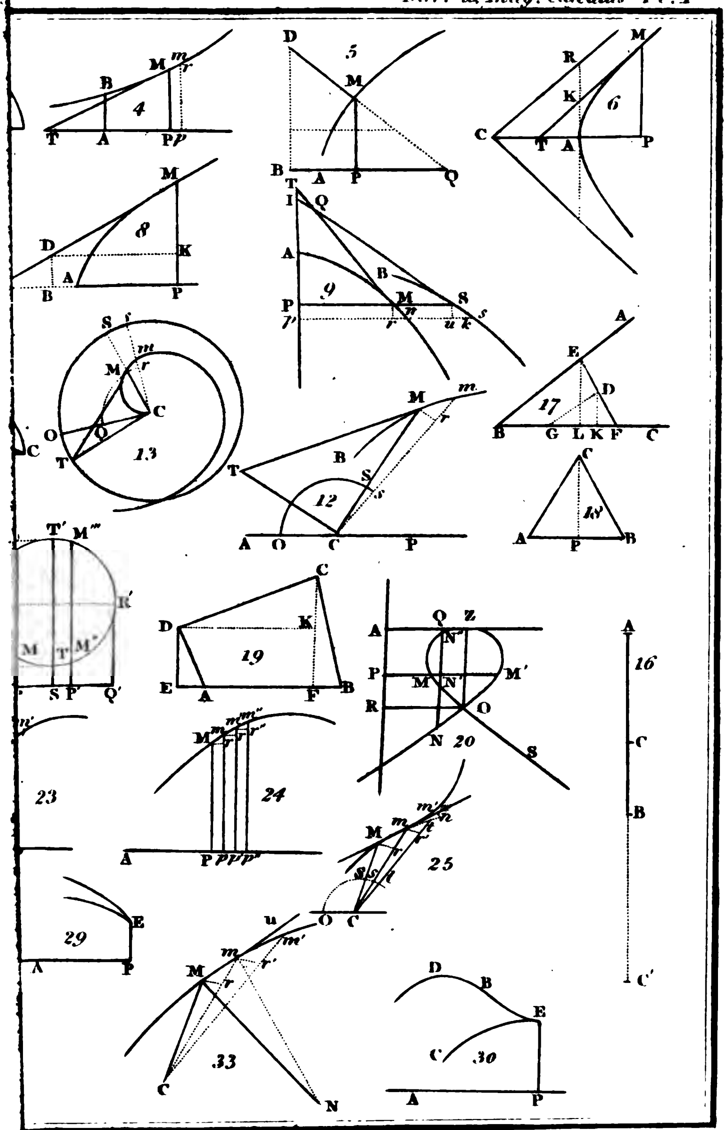

The [illegible] is a curve in which any po[illegible] [illegible] a fixed point & [illegible] [illegible] line. equation $y^2 = px$.

The Hyperbola is a curve in which the diff[illegible] of the distances drawn from any point [illegible] fixed points is [illegible]

equation $y^2 = \frac{4xc + 4c^2}{a^2}(ax + [illegible]$

The Ellipse is a curve in which the sum of the [illegible] lines drawn from [illegible] point [illegible] fixed point [illegible] given line

equation $y^2 = \frac{b^2}{a^2}(a[illegible]$

the [illegible]